Math 303, Fall 2011, Lecture 6

① The principle of Mathematical induction

Remember ω is the smallest set with $0 \in \omega$ and $n^+ \in \omega$ whenever $n \in \omega$

set of natural numbers

$\{0, \ldots, n-1\} \cup \{n\} = n \cup \{n\}$

such a set is a successor set

We saw that

ω is a subset of every successor set.

Here's another way to write that

$\boxed{\begin{array}{l}\text{If } S \text{ is a set, with} \\ \quad \bullet\ S \subseteq \omega \\ \quad \bullet\ S \text{ is a successor set} \\ \text{then } S = \omega\end{array}}$

This is called the principal of mathematical induction.

How does this relate to mathematical induction in the usual sense?

(see www.math.cornell.edu/~mec/2008-2009/ABjorndahl/ppmi.pdf
if you want a reminder on induction, or just some fun induction puzzles)

lets do an example. Suppose we want to prove inductively that

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

We will use induction on n

Base case $n=0$ left hand side is $\sum_{i=0}^{0} i = 0$

right hand side $\frac{0(1)}{2} = 0$

so result is true for $n=0$

Induction hypothesis Take $n>0$

Suppose the result is true (namely suppose $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$) for $k=n-1$

let S be the set of natural numbers n for which the statement is true

ie $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

(note $S \subseteq \omega$)

Check $0 \in S$:

same calculation

check $\sum_{i=0}^{0} i = 0 = \frac{0(0+1)}{2}$

so $0 \in S$

Suppose $(n-1) \in S$

(could use $n \in S$ and check $n^{+} \in S$ but this matches)

Inductive case: Now consider n

$$\sum_{i=0}^{n} i = \sum_{i=0}^{n-1} i + n$$

$$= \frac{(n-1)n}{2} + n$$

$$= \frac{n^2 - n + 2n}{2} = \frac{n^2+n}{2}$$

$$= \frac{n(n+1)}{2}$$

which is what we wanted to show

Conclusion: By induction

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2} \quad \text{for all } n \geq 0$$

Want to show $(n-1)^+ \in S$
$\quad\quad\quad\quad\quad\quad\quad\quad\;\; \shortparallel$
$\quad\quad\quad\quad\quad\quad\quad\quad\;\; n$

(again same computation)

$$\sum_{i=0}^{n} i = \sum_{i=0}^{n-1} i + n$$

since $n-1 \in S$

$$= \frac{(n-1)n}{2} + n$$

$$= \frac{n(n+1)}{2}$$

so $n \in S$

Conclude $S \subseteq \omega$

$0 \in S$

if $(n-1) \in S$ then $(n-1)^+ \in S$

so by the principal of mathematical induction

$S = \omega$ so

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2} \quad \text{for all natural numbers } n.$$

Last time we left some properties of natural numbers to prove once we had induction

1 No natural number is a subset of any of its elements $(*)$

proof Let S be the set of natural numbers n for which $(*)$ is true

ie $n \not\subseteq x$ for every $x \in n$

Now show $0 \in S$: 0 has no elements so certainly it is not a subset of any of its elements

Now take $n \in S$

Want to show $n^+ \in S$: $n^+ = n \cup \{n\}$. Take $x \in n^+$ (the x for n^+)

two cases (1) $x = n$ Note $n \in n^+$

if $n^+ \subseteq x$

Then $n \in x = n$ the x for n

Further $n \subseteq n \in n$

which contradicts the induction hypothesis

(2) $x \in n$ so x can also

play the role of x for n

If $n^+ \subseteq x$. Know $n \subseteq n^+$

so $n \subseteq x \in n$ which

contradicts the induction hypothesis

So taking these cases together n^+ is not a subset of any of its elements so $n^+ \in S$

By the principal of mathematical induction $S = \omega$

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2 $\boxed{\text{Let } n \text{ be a natural number, if } x \in n \text{ then } x \subseteq n}$

proof Let S be the set of natural numbers n with the property that if $x \in n$ then $x \subseteq n$

If $n = 0$. 0 has no elements so trivially if $x \in n$ then $x \subseteq n$. So $0 \in S$

Now suppose $n \in S$

Take $n^+ = n \cup \{n\}$

take $x \in n^+$ There are 2 cases

1 $x = n$ then $x \subseteq n^+$

② $x \in n$ Then by the induction hyp $x \subseteq n$

but $n \subseteq n^+$ so $x \subseteq n^+$

So by the principle of mathematical induction $S = \omega$.

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③ Let m and n be natural numbers

If $n^+ = m^+$ then $n = m$

proof Suppose $n^+ = m^+$

$$\underset{n \cup \{n\}}{\overset{n^+}{\|}} = \underset{m \cup \{m\}}{\overset{m^+}{\|}}$$

so $n \in n^+$ so $n \in m^+$

so $n = m$

or $n \in m$ so $n \subseteq m$

But can make the same argument for $m \in m^+$

either $n = m$ or $m \subseteq n$

if $n = m$ done, otherwise $n \subseteq m$ and $m \subseteq n$

so $n = m$

So in all cases $n = m$

What is wrong with the following proof

Claim All horses are the same colour (This is due to Pólya)

proof Rephrase this as for all sets of n horses all horses in the set are the same colour

If we can show this for all $n \in \omega$ we get our claim.

Base case $n=1$

I have a set of one horse
trivially all horses in the set are the same colour

Assume the result holds for sets of size $k \geq 1$

Now consider a set of $k+1$ horses.

Let $\mathcal{S}$ be a set of $k+1$ horses

Let $A, B \in \mathcal{S}$

$\mathcal{S} \setminus A$ has size k

Pick one horse from the set, call it A, the remaining horses are all the same colour (by the inductive hypothesis).

What about A? Pick another horse $B \neq A$

$\mathcal{S}-B$ has size k

so $\mathcal{S}-A$ and $\mathcal{S}-B$

are both sets with

all elements the same colour

removing B we also get a set of k horses so they're all the same colour but A is in this set so A has the same colour as the remaining ones, so did B so they are all the same colour.

so A and B have the same colour as the rest $\mathcal{S}-\{A,B\}$ so all have the same colour

Where is the problem? (this is due to Pólya)

The problem is if $\mathcal{S}-A$ and $\mathcal{S}-B$ don't overlap then can't conclude that just because each of them is all one colour then $\mathcal{S}$ is all one colour

If $k=2$ $\mathcal{S}=\{A,B\}$ so $\mathcal{S}-A=\{B\}$

$\mathcal{S}-B=\{A\}$

A trickier example:

The unexpected examination (hanging, tiger)

Suppose A teacher says Next week there will be an unexpected exam. No student will be able to predict the day of the exam until it occurs.

a Student says the exam can't be on Friday because by the time Monday through thursday have passed we'll all know the exam must be

~~M~~ ~~T~~ ~~W~~ ~~Θ~~ ~~F~~

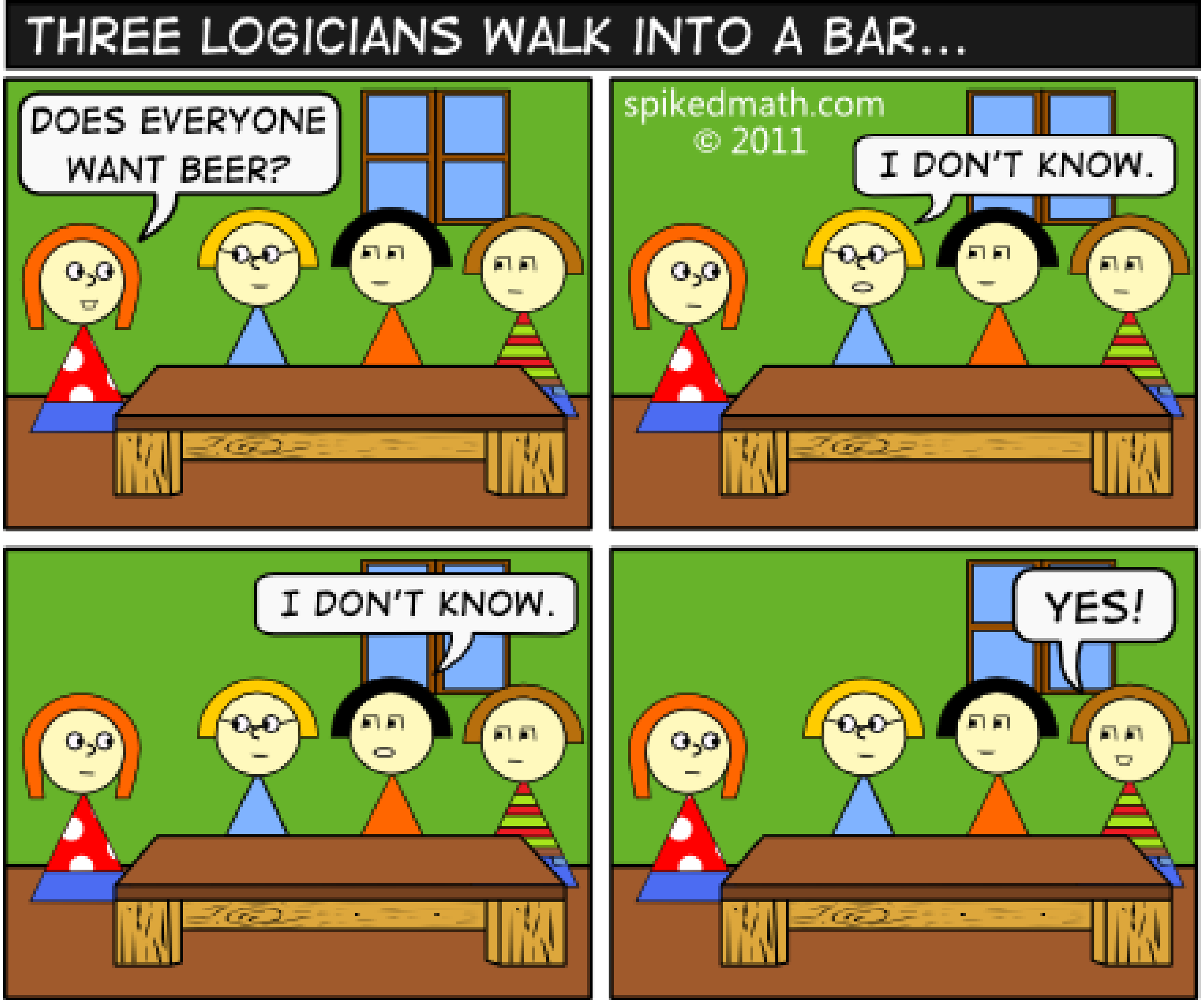
Friday so it won't be unexpected

Then the student realizes the exam can't be on thursday, because with friday ruled out by the time monday through wed. have occured we'll know it is on thursday so it won't be unexpected

By the same argument we can rule out Wed
... Tues ... Mon
So the student concludes there is no exam.

So the student is surprised by a Wednesday exam.
so it was unexpected in the end.

What is wrong with the student's argument?



(2) Next time

Functions

finite and infinite

Please read Halmos sections 7 and 8

$\underline{n} = \{0, 1, \ldots, n-1\}$

$\underline{1} = \{0\}$

if $x \in \underline{1}$ then $x = 0$

$0 = \emptyset \subseteq \{0\} = \underline{1}$

so $x \subseteq \underline{1}$