Math 303, Fall 2011, Lecture 7

(1) Functions

Suppose X and Y are sets. We would like to encode functions

$$f : X \to Y \qquad \text{using sets}$$

How can we do this?

To make this a precise definition we need

Which subsets S of $X \times Y$ are functions?

$$\boxed{\text{Define} \quad Y^X = \{ f \in \mathcal{P}(X \times Y) \mid f \text{ is a function} \}}$$

eg let $X = Y = \omega \ (= \{0, 1, 2, \dots\})$

let $f(x) = x^2$

What is f as a set?

eg let $X = \{1, 2, 3\}$ $\quad Y = \{800, 2\}$

eg let $X = \emptyset$, let $Y = \omega$

What can f be?

eg What is $Y^{\emptyset}$ for any set Y

eg let $X = \{1, 2\}$ let $Y = \{3, 4\}$

what is Y^X ?

lets remember some function words

domain $\quad \mathrm{dom} f =$

range $\quad \mathrm{ran} f =$

If onto Y

If

inclusion map

The

identity map

The

the projection map onto the second coordinate the projection map onto the first coordinate

If

one-to-one

written in function language

written in set language

eg Let $X = \{1, 2\}$ $Y = \{1, 3\}$

What is projection from $X \times Y$ to Y?

eg is the projection in the previous example onto Y?

is it one-to-one?

② Finite and infinite (This is back to Halmos p52)

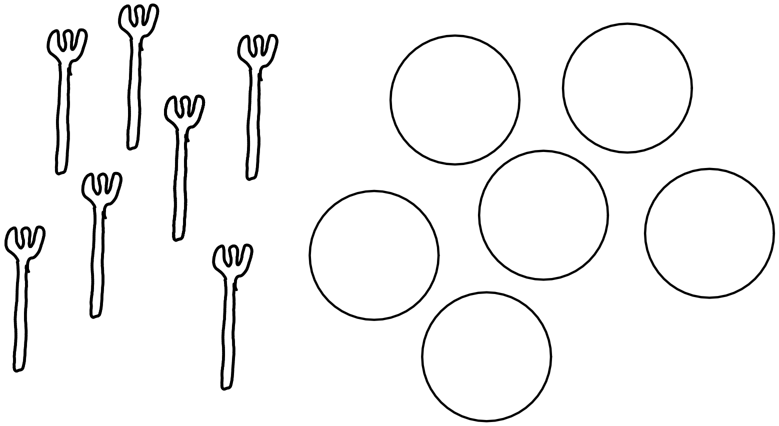
Definition Say two sets A and B are equivalent or the same size if there is a one-to-one and onto function from A to B

one-to-one correspondence

eg is $\{2, 3, 8, 12\}$ equivalent to 4?

The idea here is

eg



For the break

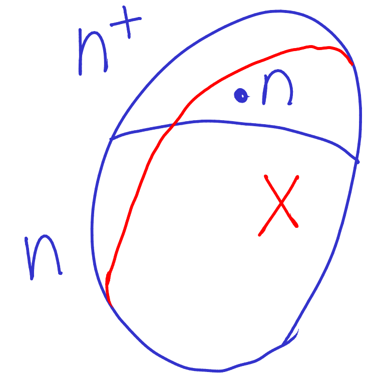
find an example of a set which is equivalent to a proper subset of itself eg

Fortunately this doesn't happen for individual natural numbers

Claim: If $n \in \omega$ then n is not equivalent to a proper subset of n

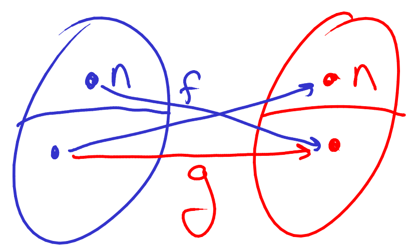
proof By induction:

if $n \in X$

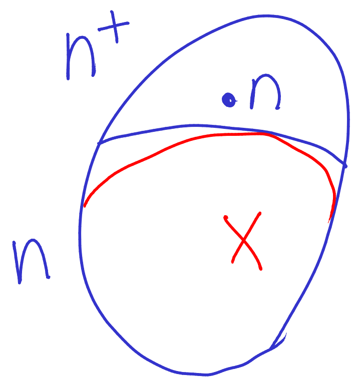


if $f(n) = n$

if $f(n) \neq n$



if $n \notin X$



Claim A set can be equivalent to at most one natural number

proof

Now we can define a set A to be finite if it is equivalent to some natural number, and infinite otherwise

Also define the size or number of elements of a finite set A to be the unique natural number equivalent to A

use the notation $\#A$ for the size of A

This notion of size corresponds to our usual notion of size

for example if $A \subseteq B$ then $\#A \leq \#B$

proof

3 Next time

- Summary of our axioms so far and outlook
- families

$\boxed{\text{Please read Halmos Section 9}}$