SIMON FRASER UNIVERSITY

DEPARTMENT OF MATHEMATICS

Midterm

Math 343 Spring 2013

Instructor: Dr. Yeats

February 21, 2013

Name: Answers (please print)

family name *given name*

SFU email: @sfu.ca

SFU-email

Signature:

Instructions:

(1) Fill in your information above.

(2) This test has 9 questions. **Complete any 8 questions.** Please indicate which question you do **NOT** want marked:

If you do not indicate anything the first answered ones will be marked.

(3) Answer in the spaces provided; use the back if necessary. Justify your answers.

(4) **No** calculators, books, papers, or electronic devices shall be within the reach of a student during the examination.

(5) **During the examination, communicating with, or deliberately exposing written papers to the view of, other examinees is forbidden.**

(1) **(5 points)** Let $\exp(x)$ be the formal power series

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Prove that

$$\exp(x)^{-1} = \exp(-x)$$

as formal power series.

It suffices to show $\exp(x)\exp(-x) = 1$ as formal power series

$$\exp(x)\exp(-x)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{x^k}{k!} \frac{(-x)^{n-k}}{(n-k)!} \right)$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k}$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} \left(\begin{cases} 1 & \text{if } n=0 \\ (1-1)^n & \text{if } n \neq 0 \end{cases} \right)$$

$= 1$ as desired

(2) We say an integer composition $n_1 + n_2 + \ldots + n_k = n$ is a *palindrome* if $n_{i+1} = n_{k-i}$ for $0 \leq i < k/2$.

(a) **(2 points)** Let $\mathcal{C}_1$ be the class of integer compositions which have an even number of parts and are palindromes. Give a specification for $\mathcal{C}_1$.

$$\mathcal{C}_1 = \mathrm{Seq}(\mathrm{Seq}_{\geq 1}(\mathcal{Z}^2))$$

where the two copies of $\mathcal{Z}$ are split between the matching parts

(or $\mathcal{C}_1 = \mathrm{Seq}_{\geq 1}(\mathrm{Seq}_{\geq 1}(\mathcal{Z}^2))$ if you don't want to allow an empty composition)

(b) **(2 points)** Let $\mathcal{C}$ be the class of all integer compositions which are palindromes. Give a specification for $\mathcal{C}$.

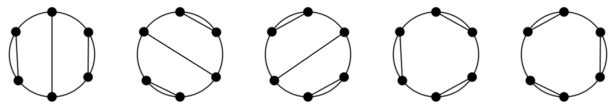
$$\mathcal{C} = \mathrm{Seq}(\mathrm{Seq}_{\geq 1}(\mathcal{Z}^2)) + \mathrm{Seq}_{\geq 1}(\mathcal{Z}) \times \mathrm{Seq}(\mathrm{Seq}_{\geq 1}(\mathcal{Z}^2))$$

(c) **(1 point)** Give an expression for the generating function of $\mathcal{C}$.

$$C(x) = \frac{1}{1 - \left(\frac{x^2}{1-x^2}\right)} + \frac{\frac{x}{1-x}}{1 - \left(\frac{x^2}{1-x^2}\right)}$$

$$= \frac{1 - x^2 + x(1+x)}{1 - x^2 - x^2} = \frac{1+x}{1-2x}$$

(3) Let $\mathcal{N}$ be the following combinatorial class. The objects of size n of this class are circles with $2n$ points and n chords joining the points so that no two chords cross or share an endpoint. For example the objects of size 3 are



(a) **(2 points)** Fix one of the points and imagine removing the chord incident to that point. This decomposes the diagram into two halves. Using this idea give a specification for $\mathcal{N}$.

$$\mathcal{N} = \underset{\substack{\uparrow \\ \text{the removed} \\ \text{chord}}}{\mathcal{Z}} \times \underbrace{(\mathcal{N} + \mathcal{E})^2}_{\leftarrow \text{the two halves}}$$

(b) **(2 points)** Find $[x^n]N(x)$.

$$N(x) = x(N(x)+1)^2 = xN(x)^2 + 2xN(x) + x$$

so $\quad xN(x)^2 + (2x-1)N(x) + x = 0$

so $\quad N(x) = \dfrac{1-2x-\sqrt{(2x-1)^2-4x^2}}{2x} = \dfrac{1-2x-\sqrt{1-4x}}{2x}$

where we chose the negative root to cancel the 1, as in lecture.

We know from lecture $[x^n]\left(-\frac{1}{2}\sqrt{1-4x}\right) = \frac{1}{n}\binom{2n-2}{n-1}$ for $n \geq 1$

and $[x^0]\left(-\frac{1}{2}\sqrt{1-4x}\right) = -\frac{1}{2}$

So $[x^n]N(x) = \begin{cases} -\frac{2}{2} + \frac{1}{1}\binom{0}{0} = 0 & \text{if } n = 0 \\ \frac{1}{n+1}\binom{2n}{n} & \text{if } n \geq 1 \end{cases}$

(c) **(1 point)** What is another combinatorial class we've seen where these same numbers appear?

Those same coefficients appeared for binary rooted trees with distinct left and right children.

(4) (a) **(2 points)** Let $F(x)$ be a formal Laurent series. Prove that

$$[x^{-1}]\frac{d}{dx}F(x) = 0$$

Write $F(x) = \sum_{i=-I}^{\infty} f_i x^i$

then $\frac{d}{dx}F(x) = \sum_{i=-I}^{\infty} i f_i x^{i-1}$

so $[x^{-1}]\frac{d}{dx}F(x) = 0 f_0 x^{-1} = 0$

(b) **(3 points)** Let k be an integer with $k \neq -1$. Let $G(x)$ be a formal Laurent series. Prove that

$$[x^{-1}]G(x)^k\frac{d}{dx}G(x) = 0$$

$G(x)^k \frac{d}{dx}G(x) = \frac{1}{k+1}\frac{d}{dx}G(x)^{k+1}$ since $k \neq -1$

so $[x^{-1}]G(x)^k\frac{d}{dx}G(x) = \frac{1}{k+1}[x^{-1}]\left(\frac{d}{dx}G(x)^{k+1}\right)$

$= 0$ by part a.

(5) Let $\mathcal{T}$ be the class of rooted trees with red and blue vertices where each red vertex has an even number of ~~blue~~ red children and an odd number of ~~red~~ blue children, and each blue vertex has at most 4 children of any colour.

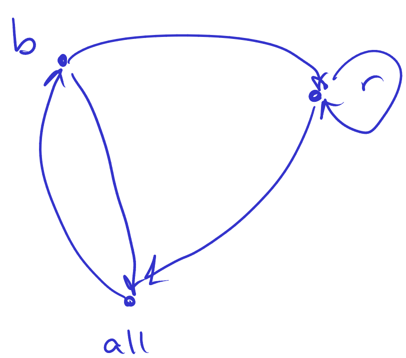
(a) **(3 points)** Give a specification for $\mathcal{T}$

$$\mathcal{T}_b = \mathcal{Z}_b \times \mathrm{Seq}_{\le 4} \mathcal{T}$$

$$\mathcal{T}_r = \mathcal{Z}_r \times \mathrm{Seq}(\mathcal{T}_r^2) \times \mathcal{T}_b \times \mathrm{Seq}(\mathcal{T}_b^2)$$

$$\mathcal{T} = \mathcal{T}_b + \mathcal{T}_r$$

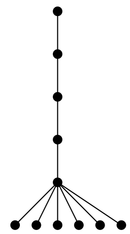
(b) **(1 point)** What is the dependency digraph of your specification?



(c) **(1 point)** What can you say about the colours of the leaves of a tree in $\mathcal{T}$?

All the leaves are blue because red vertices have an odd number of blue children and so in particular they have at least one child and so are not leaves.

(6) Let $\mathcal{B}$ be the combinatorial class of rooted trees where exactly one vertex has more than 1 child, and all the children of this vertex are leaves. Such trees look like brooms.



(a) **(3 points)** Show that $\mathcal{B}$ is regular.

We can specify

$$\mathcal{B} = \underbrace{\mathrm{Seq}_{\geq 1}(\mathcal{Z})}_{\text{the handle including the vertex with } >1 \text{ child}} \times \underbrace{\mathrm{Seq}_{\geq 2}(\mathcal{Z})}_{\text{the bristles}}$$

$$= \mathcal{Z} \times \mathrm{Seq}(\mathcal{Z}) \times \mathcal{Z}^2 \times \mathrm{Seq}(\mathcal{Z})$$

which involves only $\mathcal{Z}$, $\times$, and Seq and so is regular

(b) **(2 points)** Give a class of words with the same counting sequence as $\mathcal{B}$.

We can take binary words which begin by $\underbrace{\text{at least one } 0}_{\text{the handle}}$ and then have $\underbrace{\text{at least two 1s}}_{\text{the bristles}}$

and nothing else

so $\mathcal{W} = \mathrm{Seq}_{\geq 1}(\mathcal{Z}_0) \times \mathrm{Seq}_{\geq 2}(\mathcal{Z}_1)$

and so $B(x) = \dfrac{x^3}{(1-x)^2}$, $W(x) = \dfrac{x^3}{(1-x)^2}$

so both have the same generating function and hence the same counting sequence.

(7) **(5 points)** Rank the subsets of size k of $\{1, 2, \ldots, n\}$ by listing the elements of each subset in decreasing order and then ordering the subsets lexicographically. As in class call this the corank. Let $L = (n_k, n_{k-1}, \ldots, n_1)$ be such a subset in decreasing order. Prove that

$$\operatorname{corank}(L) = \sum_{i=1}^{k} \binom{n_{k-i+1} - 1}{k - i + 1}$$

Consider those lists of length k with elements in decreasing order which begin with

$n_k, n_{k-1} \cdots n_{k-i+2}$ and have their next element (and hence all subsequent elements) $< n_{k+i+1}$

In such a list there are $k-i+1$ spots to fill and $n_{k+i-1} - 1$ numbers to choose from

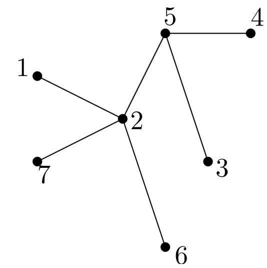
The order we choose the numbers doesn't matter since we will put them in decreasing order

Thus there are $\binom{n_{k-i+1} - 1}{k-i+1}$ such lists.

Note also that all these lists come before L in corank and every list coming before L in corank is of this form for some i.

$$\therefore \operatorname{corank}(L) = \sum_{i=1}^{k} \binom{n_{k-i+1} - 1}{k-i+1}$$

(8) **(5 points)** Find the list given by the Prüfer correspondence applied to the tree



$E = \{ \{1,2\}, \cancel{\{2,5\}}, \cancel{\{2,6\}}, \cancel{\{2,7\}}, \cancel{\{3,5\}}, \cancel{\{4,5\}} \}$

$d = (1, \cancel{4}\,\cancel{3}\,\cancel{2}\,1, \cancel{1}\,0, \cancel{1}\,0, \cancel{3}\,\cancel{2}\,\cancel{1}\,0, \cancel{1}\,0, \cancel{1}\,0)$

$i=1:\quad x=7 \quad y=2 \qquad L(1)=2$

$i=2:\quad x=6 \quad y=2 \qquad L(2)=2$

$i=3:\quad x=4 \quad y=5 \qquad L(3)=5$

$i=4:\quad x=3 \quad y=5 \qquad L(4)=5$

$i=5:\quad x=5 \quad y=2 \qquad L(5)=2$

so $\quad L = (2,2,5,5,2)$

(9) **(5 points)** Given a permutation

$$\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$$

represent σ is as the list of its values

$$(\sigma(1), \sigma(2), \ldots, \sigma(n))$$

Order the permutations of $\{1, 2, \ldots, n\}$ by ordering these lists lexicographically. Write pseudocode to describe an algorithm which takes n and a permutation σ of size n and returns the next permutation in this order.

See homework 3 solutions