COMMUTATIVE ALGEBRA, FALL 2013

ASSIGNMENT 2

(1) When I is the trivial order, the there is no compatibility condition because there is no $i \leq j \in I$. Thus being a compatible system just means being a family $f_i : A_i \to C$. This is exactly the information that a coproduct comes with. The direct limit in this case is the particular compatible system $\nu_i : A_i \to \varinjlim A_i$ which satisfies the following universal property

$$\begin{tikzcd} A_i \arrow[d, "\nu_i"] \arrow[dr, "f_i"] & \\ \varinjlim A_i \arrow[r, "f"'] & C \end{tikzcd}$$

which is the universal property for coproducts. Thus in this case the direct limit is the coproduct.

(2) Consider

$$M = \bigoplus M_i / \sum_{i \leq j} (\nu_j \phi_i^j - \nu_i) M_i$$

Let ν_i be the composition of the canonical inclusion $M_i \to \bigoplus M_i$ with the quotient map $\bigoplus M_i \to M$. So $\nu_i : M_i \to M$.

Suppose $i \leq j$. For any $m_i \in M_i$, $\nu_j(\phi_i^j(m_i)) - \nu_i(m_i)$ is 0 in M so

$$\begin{tikzcd} M_i \arrow[d, "\phi_i^j"] \arrow[dr, "\nu_i"] & \\ M_j \arrow[r, "\nu_j"'] & M \end{tikzcd}$$

commutes, thus we have a compatible system. Now suppose we have another compatible system $f_i : M_i \to C$. Define $f : M \to C$ by $f(\sum m_i) = \sum f_i(m_i)$. This is the only possibility which could give the map in the universal property because the universal property gives that $f(\nu_i(m_i)) = f_i(m_i)$ for all $m_i \in M_i$.

It remains to check that f is well-defined. Take some $m_i \in M_i$ and $i \leq j$ then let $m = \nu_j(\phi_i^j(m_i)) - \nu_i(m_i)$. Then

$$f(m) = f(\phi_j^i(m_i)) - f(m_i) = 0$$

since the f_i with C form a compatible system. Thus

$$\sum_{i \leq j} (\nu_i(\phi_i^j - \nu_i) M_i \subseteq \ker(f)$$

and hence f is well-defined. Therefore M is the direct limit in modules.

(3) Suppose I is directed. For $a \in_i$ and $b \in A_j$ write $a \sim b$ if $\phi_i^k a = \phi_j^k b$ for some $k \geq i, j$. Let

$$A = \bigcup A_i / \sim$$

We need to show that A is an object of the category. The objects of the category are those sets with some given relations and functions which satisfy certain first order axioms. So proceed by induction on formulas.

Consider a quantifier free formula ϕ and a particular assignment of elements of $\bigcup A_i$ to the free variables of ϕ. There are only finitely many variables in ϕ, so take an upper bound $A_j$of the A_i the elements assigned to them appear. By compatibility, in A_j and every set above it in the order ϕ with this assignment is either true (in all of them) or false (in all of them). By definition of $\sim$, ϕ with the assignment is true in A_j iff it is true in A.

Consider a formula of the form $\exists x \phi(x)$ and an assignment of elements of $\bigcup A_i$ to the free variables of $\exists x \phi(x)$. By induction any assignement of an element to x in $\phi(x)$ is simultaneously true in A and in all sufficiently high A_j in the order or simultaneously false in these sets. If there is some such assignment for x making $\phi(x)$ true then $\exists x \phi(x)$ is simultaneously true in A and in all sufficiently high A_j.

Consider a formula of the form $\forall x \phi(x)$ and an assignment of elements of $\bigcup A_i$ to the free variables of $\forall x \phi(x)$. By induction any assignement of an element to x in $\phi(x)$ is simultaneously true in A and in all sufficiently high A_j in the order or simultaneously false in these sets. If all such assignment for x from a sufficiently high A_j make $\phi(x)$ true then $\forall x \phi(x)$ is simultaneously true in A and in all sufficiently high A_j.

Thus we conclude that the axioms of the category, which must be true in all A_j, must also be true in A. So A is an object in the category.

Let $\nu_i : A_i \to A$ be the natural map $A_i \to \bigcup A_i$ composed with the quotient map $\bigcup A_i \to A$. Next we need to show that A along with the ν_i is a direct limit. This works very much like the previous question.

Compatiblity holds because for $a_i \in A_i$,

$$\phi_i^j(a_i)) = \phi_j^j(\phi_i^j(a_i))$$

so $a_i \sim \phi_i^j(a_i)$ and so $\nu_j(\phi_i^j(a_i)) = \nu_i(a_i)$ in A.

Supppose we have another compatible system $f_i : A_i \to C$. Then define $f : A \to C$ by $f(a_i) = f_i(a_i)$ for $a_i \in A$. This identity is required by the diagram, and so this is the only possible map. The remaining question if is f is well defined.

To see that f is well defined suppose we have $\phi_i^k a = \phi_j^k b$ with $a \in A_i$ and $b \in A_j$. Then $f(a) = f_i(a)$ and $f(b) = f_j(b)$. But by compatibility

$$f_i(a) = f_k(\phi_i^k(a)) = f_k(\phi_j^k(b)) = f_j(b).$$

Therefore A is the direct limit in this category.

Finally, consider the case where I is a chain. Then $\sim$ reduces to $a \sim b$ if $\phi_i^j(a) = b$, so we are in effect identifying a with its image in all higher sets in the chain. If the chain has an upper bound, call it $A_{\max}$ then this tells us that $(\bigcup A_i / \sim) \cong A_{\max}$.

(4) This is easier than the previous one since we aren't asked to show the inverse limit exists in the given category.

Let
$$A = \{(a_i) \in \prod A_i : a_i = \phi_i^j a_j \forall i \leq j\}$$
Assume A exists in the the given category. Let $\nu_i : A \to A_i$ be the restriction of the projection $\prod A_i \to A_i$ to A.

This gives a compatible system as if $a_i \in A_i$ and $i \leq j$ then $\nu_i(a_i) = \phi_i^j \nu_j(a_j)$ in A by construction.

Suppose we have another compatible system $g_i : C \to A_i$. Define $f : C \to A$ by $f(c) = (g_i(c))$. This is the only possible map since the universal property diagram requires $\nu_i(f(c)) = g_i(c)$. Finally f is well defined since by compatibility $\phi_i^j g_j(c) = g_i(c)$.

Thus A is the inverse limit.

(5) Let U be the forgetful functor from groups to sets. We need a natural transformation τ such that

$$\begin{array}{ccc} \mathrm{Hom}(\mathbb{Z}, G) & \xrightarrow{\tau_G} & UG \\ \Big\downarrow f_* & & \Big\downarrow Uf \\ \mathrm{Hom}(\mathbb{Z}, H) & \xrightarrow{\tau_H} & UH \end{array}$$

for any group homomorphism $f : G \to H$, and such that each τ_G is a set bijection. Note that any $h \in \mathrm{Hom}(\mathbb{Z}, G)$ is determined by where it sends 1, and that since $\mathbb{Z}$ is freely generated by 1, any value for $h(1)$ is valid. So define $\tau_G : \mathrm{Hom}(\mathbb{Z}, G) \to UG$ by $\tau_G(h) = h(1)$. This is a set bijection by the above observations. The required diagram commutes because $f(\tau_G(h)) = f(h(1)) = f_*(h)(1) = \tau_H(f_*(h))$. Thus the forgetful functor from groups to sets is represented by $\mathbb{Z}$.

(6) As many of you noticed there is a problem with (vi) because it doesn't say anything about how K depends on N, and hence it only tells us (dually to Proposition 2.8) that K is a direct summand, not that it is the matching one for N. There are a number of additional assumptions you could add to (vi) to clear this up. I'm not sure which one is tidiest. I will add to (vi) that $0 \to K \to M \to N \to 0$ is exact.

Now to solving the problem. First note that $(i) \Leftrightarrow (iv)$ is Proposition 2.8. Also $(v) \Rightarrow (vi)$ and $(v) \Rightarrow (iv)$ are both trivial since (vi) and (iv) are (a priori) weakenings of (v).

$(i) \Rightarrow (iii)$: Write $M = K \oplus N$. Let $\pi : M \to N$ be the identity restricted to N and 0 restricted to M. This is well defined since the sum is direct. Then $\pi(M) = N$ and $\pi^2 = \pi$.

$(iii) \Rightarrow (iv)$: Take π from (iii). Viewed as $\pi : M \to N$ then π is epic and it is split by the inclusion map of N in M.

$(iii) \Rightarrow (ii)$: Take π from (iii). If $\pi(a) = a$ then $a \in \pi(M)$ so $a \in N$. If $a \in N$ then there is a b such that $\pi(b) = a$ so $\pi(a) = \pi^2(b) = \pi(b) = a$.

$(ii) \Rightarrow (iii)$: Take π from (ii). Thus $\pi(M) \subseteq N$, and $N = \{a \in M : \pi(a) = a\}$ so $\pi(M) = N$. Also $\pi^2(b) = \pi(\pi(b)) = \pi(b)$ since $\pi(b) \in N$. so $\pi^2 = \pi$.

$(i) \Rightarrow (v)$: Write $M = K \oplus N$. Then
$$0 \to K \to K \oplus N \to N \to 0$$

is an exact sequence where the map from K to $K \oplus N$ is the natural inclusion and the map from $K \oplus N$ to N is the map which is the identity on N and is 0 on K (which as noted above is well defined by directness). Both these maps split. The splitting map from $K \oplus N$ to K is the map which is the identity on K and 0 on N, while the splitting map from N to $K \oplus N$ is the natural inclusion.

Finally we need (vi) to anything else. Which one is easiest will depend on what assumption you added to (vi). I'll do $(vi) \Rightarrow (v)$: I have already assumed we have the exact sequence and we already know ν splits. The natural inclusion splits π and so we're done.

(7) First check $\text{tor}(M)$ is a submodule of M.

Suppose $m_1, m_2 \in \text{tor}(M)$. Then there exists s_1, s_2 regular elements of R such that $s_1m_1 = 0$ and $s_2m_2 = 0$. s_1s_2 is also regular since if there were some other $r \in R$ with $s_1s_2r = 0$ then $s_1(s_2r) = 0$ so by regularity of s_1, $s_2r = 0$ and hence by regularity of s_2, $r = 0$. Also $s_1s_2(m_1 + m_2) = s_2(s_1m_1) + s_1(s_2m_2) = 0$ since R is commutative. Thus $m_1 + m_2 \in \text{tor}(M)$.

Suppose $m \in \text{tor}(M)$. Then there exists s regular in R such that $sm = 0$. Suppose also $r \in R$. Then $s(rm) = r(sm) = 0$ so $rm \in \text{tor}(M)$.

Next check the direct sum property. To keep notation easier, we can without loss of generality take the M_j disjoint and hence view $M_j \subseteq \bigoplus M_i$ for all j.

Take $\sum m_i \in \bigoplus \text{tor}(M_i)$. Say $s_im_i = 0$ with s_i regular. As the sum is finite $s = \prod s_i \in R$ and s is regular by induction on the argument used above in closure under $+$. Furthermore $s(\sum m_i) = 0$. so $\sum m_i \in \text{tor} \bigoplus M_i$. Therefore $\bigoplus \text{tor} M_i \subseteq \text{tor} \bigoplus M_i$.

Take $\sum m_i$ in $\text{tor} \bigoplus M_i$ and say $s \sum m_i = 0$ with s regular. Then $\sum sm_i = 0$ and by directness $sm_i = 0$ for all i. Therefore $\sum m_i \in \bigoplus \text{tor} M_i$. Therefore $\text{tor} \bigoplus M_i \subseteq \bigoplus \text{tor} M_i$.

All together $\text{tor} \bigoplus M_i = \bigoplus \text{tor} M_i$.

(8) For groups we need chains where the factors make sense, so we need chains of the form

$$G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_t$$

Such a chain is called a *subnormal chain*. Equivalence of chains is the same as in the module case. A composition series of groups is a subnormal chain which ends at 0 and with all quotients G_i/G_{i+1} simple. The theorem then will say that if a group has a composition series, then

- Every finite subnormal chain beginning at G can be refined to a composition series
- Every composition series of G has the same length, $\ell(G)$
- For every normal subgroup H of G, H and G/H have composition series and $\ell(G) = \ell(H) + ell(G/H)$.

(9) I didn't say anything about finitely generated so consider $M = R[\lambda_1, \lambda_2, \ldots]$ as an R module. We have the infinite chain of modules $M \supset R[\lambda_2, \lambda_3, \ldots] \supset R[\lambda_3, \lambda_4, \ldots] \supset \cdots$. If M had a composition series, say of length n, then truncate the infinite chain of modules to length $n+1$, and by the composition series theorem we would have to be able to refine this chain to a composition series of length n. This is a contradiciton. Therefore M has no composition series.