COMMUTATIVE ALGEBRA, FALL 2013

ASSIGNMENT 3 SOLUTIONS

(1) Suppose $a \in T$ and $r \in R$ with the property that $f(r)a \neq af(r)$. Consider then $\hat{f}(\lambda)\hat{(f)}(r)$. By equation 5.2 this would be $af(r)$, but for $\hat{f}$ to be a homomorphism it must also equal $\hat{f}(r\lambda)$ (since λ is a commuting indeterminate), which by equation 5.2 is $f(r)a$. Buit $f(r)a \neq af(r)$ so equation 5.2 does not define a homomorphism.

(2) Let R be an affine algebra. Write $R = F[\lambda_1, \ldots, \lambda_n]/A$ for A some ideal of $F[\lambda_1, \ldots, \lambda_n]$. $F[\lambda_1, \ldots, \lambda_n]$ has a basis

$$\{\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n, \lambda_1^2, \lambda_1\lambda_2, \lambda_1\lambda_3, \ldots, \lambda_2^2 \ldots, \lambda_1^3, \ldots\}$$

which is countable (first list all the monomials of degree 1, then all the monomials of degree 2, and so on). $R = F[\lambda_1, \ldots, \lambda_n]/A$ is a subspace and so also has countable dimension.

(3) Suppose

$$\sum_{i=1}^{n} f_i \left(\frac{1}{\lambda - \alpha_i}\right) = 0$$

for some $f_i \in F$. Then clearing denominators we get

$$\sum_{i=1}^{n} f_i \prod_{i \neq j} (\lambda - \alpha_i) = 0$$

Now specialize λ to α_1 to get $f_1 = 0$. Similarly for the other values of i we get $f_i = 0$ for all i. Thus the original set is linearly independent over F and so $[F(\lambda) : F] \geq |F|$.

(4) Let $i = \sqrt{-1}$. $\mathbb{Z}[i]$ is integral over $\mathbb{Z}$ since i satisfies $\lambda^2 + 1 = 0$.

Observe that the norm (absolute value) of a product of complex numbers is the product of their norms and if $|a + bi| = 1$ with $a, b \in \mathbb{Z}$, then $a^2 + b^2 = 1$ so $a + bi \in \{1, -1, i, -i\}$ which are the units of $\mathbb{Z}[i]$.

The elements of norm 5 in $\mathbb{Z}[i]$ satisfy $|a+bi| = 5$ so $a^2+b^2 = 5$ so $\{\pm a, \pm b\} = \{1, 2\}$. By multiplying by i (a unit) if necessary we may assume that $b = \pm 1$ and $a = \pm 2$. Then multiplying by -1 if necessary we may assume $a = 2$.

Now $5 = 2^2 + 1^2 = (2 + i)(2 - i)$. Consider the ideal $P_1 = \langle 2 + i \rangle$ in $\mathbb{Z}[i]$. P_1 lies over $5\mathbb{Z}$ because if $(2 + i)(a + bi) \in \mathbb{Z}$ then $(2 + i)(a + bi) = 2a - b + (a + 2b)i \in \mathbb{Z}$, so $a + 2b = 0$, so $(2 + i)(a + bi) = 2(-2b) - b = -5b \in 5\mathbb{Z}$. P_1 is prime because $|2 + i| = 5$, so if this element is a product of two others, then one of the two must have norm 1 and hence be a unit.

Furthermore, P_1 is maximal for the following reason. If we were to adjoin any element of norm relatively prime to 5, then the norm of the gcd of these elements would be 1 and hence the ideal would contain a unit. On the other hand if we were to adjoin a multiple of $2 - i$, say $z(2 - i)$ then $(2 + i)z(2 - i) - z(2 + i) - z(2 - i) = z$

is in the ideal. Continuing likewise we can remove all powers of $2-i$ from z and then either get 1 in the ideal or an element of norm relatively prime to 5, hence again 1.

By an analogous argument $P_2 = \langle 2-i \rangle$ is also a maximal ideal of $\mathbb{Z}[i]$ which lies over $5\mathbb{Z}$, thus two different ideals lie over $5\mathbb{Z}$.

Let $P = p\mathbb{Z}$ be a prime ideal of $\mathbb{Z}$. Suppose we can write $p = a^2 + b^2$ in $\mathbb{Z}$ (which by number theory we know occurs when $p \equiv 1 \mod 4$, and the decomposition as two squares is unique up to order and signs of a and b). Then the same argument as above will give that $P_1 = \langle a + bi \rangle$ and $P_2 = \langle a - bi \rangle$ are distinct maximal ideals of $\mathbb{Z}[i]$ which lie over P.

If we cannot write $p = a^2 + b^2$ then there is no element of $\mathbb{Z}[i]$ of norm p. So any factor of p in $\mathbb{Z}[i]$ has norm 1 or p^2. Thus p is itself irreducible in $\mathbb{Z}[i]$, and hence the ideal, P' generated by p in $\mathbb{Z}[i]$ is a prime ideal lying over $p\mathbb{Z}$. P' is the only ideal lying over P because P' is generated by the generator of P which must be in any ideal lying over P.

(5) Suppose we have $\phi : C \to K$ with K algebraically closed. $\ker\phi$ is an ideal of C and so $C/\ker\phi$ is isomorphic to a subring of K and hence is an integral domain. Therefore $\ker\phi$ is a prime ideal of C.

By LO there exists a prime ideal Q of R lying over $\ker\phi$. That is $Q \cap C = \ker\phi$. Then R/Q is integral over $C/\ker\phi$ and hence is integral over K. But K is algebraically closed, so $R/Q \cong K$; call the isomorphism ψ. By construction, ψ extends the isomorphism of $C/\ker\phi$ to a subalgebra of K. Hence the map $R \to K$ given by $r \mapsto \psi(r + Q)$ gives the desired homomorphism extending ϕ.

(6) Let $P = \langle \lambda_1 - \lambda_4^3, \lambda_2 - \lambda_4^4, \lambda_3 - \lambda_4^5 \rangle$.

P is prime because $F[\lambda_1, \lambda_2, \lambda_3, \lambda_4]/P \cong F[\lambda_4]$ via $\lambda_1 \mapsto \lambda_4^3$, $\lambda_2 \mapsto \lambda_4^4$, $\lambda_3 \mapsto \lambda_4^5$, $\lambda_4 \mapsto \lambda_4$, and $F[\lambda_4]$ is an integral domain. Further $F[\lambda_4]$ is not a field, so P is not maximal.

Let $Q = P \cap C$ where $C = F[\lambda_1, \lambda_2, \lambda_3]$. Then Q is a prime ideal of C. Note that $\lambda_1\lambda_3 - \lambda_2^2 \in Q$. Then $0 \subsetneq \langle \lambda_1\lambda_3 - \lambda_2^2 \rangle \subsetneq Q$ each of which is prime, so the height of Q is at least 2. On the other hand, the transcendence degree, and hence the Krull dimension, of C is 3. Further, Q is not maximal because P is not maximal and P lies over Q and R is integral over C. Thus the height of Q is at most $3 - 1 = 2$. Therefore the height of Q is 2.

Notice that two more elements of Q are $\lambda_3^2 - \lambda_1^2\lambda_2$ and $\lambda_1^3 - \lambda_2\lambda_3$. Next note that Q contains no elements with a (nonzero) constant term because P contains no such elements (since the generators have no constant terms so all linear combinations of them also have no constant terms).

Suppose Q has an element with a linear term. Suppose it can be written

$$p_1(\lambda_1 - \lambda_4^3) + p_2(\lambda_2 - \lambda_4^4) + p_3(\lambda_3 - \lambda_4^5)$$

with $p_i \in F[\lambda_1, \lambda_2, \lambda_3, \lambda_4]$. Let c_i be the constant term of p_i. The $-c_1\lambda_4^3$ cannot cancel with another term since all other terms are either higher degree in λ_4 or involve another variable. Thus $c_1 = 0$. For the $-c_2\lambda_4^4$ term to cancel we must have $c_2\lambda_4$ as a term in p_1. Then we also get a $c_2\lambda_1\lambda_4$ term which cannot be cancelled, and so $c_2 = 0$.

Similarly then to cancel $-c_3\lambda_4^5$ we must have $d_1\lambda_4^2$ in p_1 and $d_2\lambda_4$ in p_2 with $d_1 + d_2 = c_3$. But then we get $d_1\lambda_4^2\lambda_1$ and $d_2\lambda_2\lambda_4$ terms neither of which can be cancelled, and so $d_1 = d_2 = 0$. This give $c_3 = 0$, and thus Q contains no element with a nonzero linear term.

Consider the homogeneous components of degree 2 of elements of Q. We know $\lambda_1\lambda_3 - \lambda_2^2$, λ_3^2, ad $\lambda_2\lambda_3$ are examples. But these three elements alone span a vector space of dimension 3, and multiplying by nonconstant polynomials only increases the degree. Therefore Q has at least 3 generators. In particular Q cannot be generated by 2 elements.

(7) Let $M_i = \{\frac{m}{n} : \gcd(m, n) = 1, n$ involves none of the first i primes$\}$. Let $N_i = M_i/\mathbb{Z}$.

$$\mathbb{Q} = M_0 \supsetneq M_1 \supsetneq M_2 \supsetneq \cdots$$

so

$$\mathbb{Q}/\mathbb{Z} = N_0 \supseteq N_1 \subseteq N_2 \supseteq \cdots$$

The only question remaining is whether the containments remain strict after modding out by $\mathbb{Z}$. Let p_i be the ith prime. Take $\frac{1}{p_i} + \mathbb{Z} \in N_{i+1}$. If $\frac{1}{p_i} + \mathbb{Z} \in N_i$ then there exists $\frac{m}{n}$ with $\gcd(n, p_i) = 1$ and $\frac{m}{n} - \frac{1}{p_i} = \ell \in \mathbb{Z}$. But then $mp_i - n = \ell n p_i$ which is impossible. Thus the containments are strict and so $\mathbb{Q}/\mathbb{Z}$ is not Artinian.

(8) Take a finitely generated submodule N of M. Then N is Noetherian since it is finitely generated, and M/N is Noetherian by hypothesis. So by a result from class M is also Noetherian.

(9) Here are a few which were bad enough that we commented on them in class. Sloppyness regarding 0 and ultrafilters in the ch 0 problems. The ideal in ch 6 exercise 9. A $\neq$ which should be an $=$ in Lemma 6.30. How many more did you find?