Results on chain complex's: The notes basically follow Rotman Advanced Modern Algebra, chapter 10 on Homology.

Definition: A chain map $f_\bullet : A_\bullet \to B_\bullet$ is null homotopic if there are maps $s_n : A_n \to B_{n+1}$ such that $f_n = s_{n-1}d_n + d'_{n+1}s_n$. In diagrams

$$\begin{tikzcd}
{} \arrow[r] & A_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] & A_n \arrow[r, "d_n"] \arrow[dl, "s_n"'] \arrow[d, "f_n"'] & A_{n-1} \arrow[r] \arrow[dl, "s_{n-1}"'] \arrow[d, "f_{n-1}"] & {} \\
{} \arrow[r] & B_{n+1} \arrow[r, "d'_{n+1}"'] & B_n \arrow[r, "d'_n"'] & B_{n-1} \arrow[r] & {}
\end{tikzcd}$$

We say that chain maps f, g are homotopic if $f - g$ is null homotopic and write $f \sim g$.

This idea comes from algebraic topology. I do not know much about the field, but the little I know suggests that this intuitively means that the maps f and g can be "continuously" deformed into the same thing.

Proposition: Let $f_\bullet, g_\bullet$ be chain maps from $C_\bullet \to B_\bullet$ with $f \sim g$. Then $H_n(f_\bullet) = H_n(g_\bullet)$.

Proof: To remind ourselves, we have the following diagram.

$$\begin{tikzcd}
{} \arrow[r] & A_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] & A_n \arrow[r, "d_n"] \arrow[dl, "s_n"'] \arrow[d, "f_n"'] & A_{n-1} \arrow[r] \arrow[dl, "s_{n-1}"'] \arrow[d, "f_{n-1}"] & {} \\
{} \arrow[r] & B_{n+1} \arrow[r, "d'_{n+1}"'] & B_n \arrow[r, "d'_n"'] & B_{n-1} \arrow[r] & {}
\end{tikzcd}$$

Let z_n be an $n-cycle$. Note that, $d_n z_n = 0$ by the definition of n cycle. So plugging into our formula gives

$$f_n z_n - g_n z_n \quad = \quad d'_{n+1}s_n z_n + s_{n-1}d_n z_n = d'_{n+1}s_n z_n \in B_n(B_\bullet)$$

That is, modulo$B_n(B_\bullet)$ we have $f_n = g_n$. But this is precisely the statement that $H_n(f_\bullet) = H_n(g_\bullet)$.

Prop: Let $C_\bullet$ be contractible, or that the identity chain map $1_{C_\bullet}$ is null homotopic. Then $C_\bullet$ is acyclic. (exact)

Proof: We have

$$\begin{tikzcd}
{} \arrow[r] & A_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "1_{n+1}"'] & A_n \arrow[r, "d_n"] \arrow[dl, "s_n"'] \arrow[d, "1_n"'] & A_{n-1} \arrow[r] \arrow[dl, "s_{n-1}"'] \arrow[d, "1_{n-1}"] & {} \\
{} \arrow[r] & A_{n+1} \arrow[r, "d_{n+1}"'] & A_n \arrow[r, "d_n"'] & A_{n-1} \arrow[r] & {}
\end{tikzcd}$$

Let z_n be an n cycle. We have to show that z_n in the boundary n boundary. We know that $1_n = s_{n-1}d_n + d_{n+1}s_n$. Then as $d_n z_n = 0$ we have

$$z_n = s_{n-1}d_n z_n + d_{n+1}s_n z_n = d_{n+1}s_n z_n \in B_n(B_\bullet)$$

This shows that $B_n(A_\bullet) = Z_n(A_\bullet)$ or that $A_\bullet$ is acyclic.

A sequence of complex's $A_\bullet \xrightarrow{i_\bullet} B_\bullet \xrightarrow{p_\bullet} C_\bullet$ is exact if for all n we have

$$\mathrm{im} i_n = \ker p_n$$

This is compact notation, note that this is really a huge diagram!

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r] & A_{n+1} \arrow[r, "d_{n+1}"'] \arrow[d, "i_{n+1}"'] & A_n \arrow[r, "d_n"'] \arrow[d, "i_n"'] & A_{n-1} \arrow[r] \arrow[d, "i_{n-1}"'] & {} \\
{} \arrow[r] & B_{n+1} \arrow[r, "d'_{n+1}"'] \arrow[d, "p_{n+1}"'] & B_n \arrow[r, "d'_n"'] \arrow[d, "p_n"'] & B_{n-1} \arrow[r] \arrow[d, "p_{n-1}"'] & {} \\
{} \arrow[r] & C_{n+1} \arrow[r, "d''_{n+1}"'] \arrow[d] & C_n \arrow[r, "d''_n"'] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & {} \\
 & {} & {} & {} &
\end{tikzcd}$$

Connecting Homomorphism

Suppose that we have a short exact sequence of complexes.

$$0_\bullet \to C'_\bullet \xrightarrow{i_\bullet} C_\bullet \xrightarrow{p_\bullet} C''_\bullet \to 0_\bullet$$

Then for all n we have a homomorphism

$$\partial_n : H_n(C''_\bullet) \quad \to \quad H_{n-1}(C'_\bullet)$$

given by

$$\partial_n(z_n + B_n C'') = i_{n-1}^{-1} d_n p_n^{-1} z_n + B_n(C')$$

(The lifting here is ambiguous, but it turns out that such a choice is unique)

This is just a diagram chase. It is routine. In all honesty, I have found reading diagram chases to be pointless without doing them. In my own experience, these things become transparent when you just do them. The proof is in Rotman in any case.

Long Exact Sequence

Let

$$0_\bullet \to C'_\bullet \xrightarrow{i_\bullet} C_\bullet \xrightarrow{p_\bullet} C''_\bullet \to 0_\bullet$$

be a exact sequence of complexes. Then there is a long exact sequence

$$\to H_{n+1}(C''_\bullet) \xrightarrow{\partial_{n+1}} H_n(C'_\bullet) \xrightarrow{i_*} H_n(C_\bullet) \xrightarrow{p_*} H_n(C''_\bullet) \xrightarrow{\partial_n} H_{n-1}(C'_\bullet) \to$$

Proof:Its a diagram chase. As before the proof is in Rotman.

This gives a nice commutative diagram

$$\begin{tikzcd}
H_n(C'_\bullet) \arrow[dr, "i_*"] & \\
H_n(C''_\bullet) \arrow[u, "\partial"] & H_n(C_\bullet) \arrow[l, "p_*"]
\end{tikzcd}$$

Lets talk about Ext baby

We start with a definition.

Definition: Let M be an R module and let

$$.... \quad \to P_2 \to P_1 \xrightarrow{d_1} \quad P_0 \to M$$

be a projective resolution.

The Corresponding deleted projective resolution of M is

$$.... \to P_2 \to P_1 \xrightarrow{d_1} P_0 \to 0$$

We can always recover M from the deleted projective resolution since $M \cong \text{coker} d_1$.

Definition: Let M and N be R modules and take a deleted projective resolution of M say $\mathcal{P}$. Now apply the contra variant hom functor $Hom_R(-, N)$ to reverse all the arrows for $\mathcal{P}$. So we get

$$0 \to \hom(P_0, N) \to \hom(P_1, N) \to ...$$

But this does not really change anything. The standard convention is to use super scripts now as the indexing is increasing.

Then we define

$$Ext^n_R(M, N) \quad = \quad H^n(\hom_R(\mathcal{P}, N))$$

We still need to check that this is independent of choice of projective resolution! However, it is independent of the injective resolutions and in fact and different resolutions give naturally isomorphic results.

We now run through some results about Ext.

But first, a lemma.

If

$$A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

is an exact sequence of R modules. Then given an R module D we have that that

$$0 \to \hom(A, D) \xrightarrow{i^*} \hom(B, D) \xrightarrow{p^*} \hom(C, D)$$

is also an exact sequence. In fact the above can be strengthened to be an if and only if. The proof can be found in say Hungerfords algebra along with similar statements. In fact he has a whole section on hom and projective/injective modules which is pretty good.

Prop: What is $Ext^0_R(M, N) \cong \hom_R(M, N)$.

Pf: By definition, we know that $Ext^0_R(M, N) = H^0(\hom_R(\mathcal{P}, N))$

Now take a projective resolution and look at the end

$$P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$$

Taking homs gives

$$0 \to \hom(M, N) \xrightarrow{d_0^*} \hom(P_0, N) \xrightarrow{d_1^*} \hom(P_1, N)$$

By the above lemma we have that the sequence is exact. The sequence we are interested in is

$$0 \to \hom(P_0, N) \xrightarrow{d_1^*} \hom(P_1, N) \to ...$$

So $Ext^0_R(M,N) \cong \ker d_1^*$. But by the exactness of the first sequence we have that

$$\ker d_1^* = \mathrm{im} d_0^* \cong \hom(M,N)$$

Lemma: Horseshoe Lemma: Given a diagram with the bottom row exact and columns projective resolutions

$$\begin{tikzcd}
 & \arrow[d] & & \arrow[d] & \\
 & P_1' \arrow[d] & & P_1'' \arrow[d] & \\
 & P_0' \arrow[d, "\epsilon'"'] & & P_0'' \arrow[d, "\epsilon''"'] & \\
0 \arrow[r] & A' \arrow[r, "i"'] & A \arrow[r, "p"'] & A'' \arrow[r] & 0
\end{tikzcd}$$

There is a projective resolution of A and chain maps so the above diagram can be completed to a exact sequence of chain maps.

Proof: Put $P_0 := P_0' \oplus P_0''$. Since sums of projective modules are projective, P_0 is projective. Let i_0 be the canonical injection $P_0' \to P_0$ and p_0 the canonical projection from P_0 to P_0''.

So we get an exact sequence

$$0 \to P_0' \to P_0 \to P_0'' \to 0$$

Now, as P_0'' is projective there is some $f : P_0'' \to A$ with $\epsilon'' = pf$. So if π', π'' are the canonical projections of P_0 we can define $\epsilon : P_0 \to A$ by

$$\epsilon \quad = \quad i\epsilon'\pi' + f\pi''$$

Now set $K_0' = \ker \epsilon'$, $K_0'' = \ker \epsilon''$ and $K_0 = \ker \epsilon$. So we obtain a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K_0' \arrow[r] \arrow[d] & K_0 \arrow[r] \arrow[d] & K_0'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0' \arrow[r, "i_0"'] \arrow[d, "\epsilon'"'] & P_0 \arrow[r, "p_0"'] \arrow[d, "\epsilon"'] & P_0'' \arrow[r] \arrow[d, "\epsilon''"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "i"'] \arrow[d] & A \arrow[r, "p"'] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Where the top row is given by the maps that come from the snake lemma. Now, by the 3 lemma or by the long exact sequence the whole thing must be exact.

We now show by induction how to proceed on the diagram below. Note, that we have shown that we can construct the first step up in the projective resolution of A.

Now suppose that we have the first n rows filled up. Put $K_n := \ker(P_n \to P_{n-1})$ and define K'_n and K''_n in the obvious way. So we obtain a new 3×3 diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K'_n \arrow[r] \arrow[d] & K_n \arrow[r] \arrow[d] & K''_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P'_n \arrow[r] \arrow[d] & P_n \arrow[r] \arrow[d] & P''_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P'_{n-1} \arrow[r] \arrow[d] & P_{n-1} \arrow[r] \arrow[d] & P''_{n-1} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Now that since the two rows on the left and right of the diagram are exact, that the map $P'_{n+1} \to P'_n$ factors through K'_n that makes everything commute, and similarly with $P''_{n+1} \to P''_n$ and K''_n. Define $P_{n+1} := P'_{n+1} \oplus P''_{n+1}$ and mimic what we did above with the original diagram to obtain a new diagram that commutes and has exact rows

$$
\begin{tikzcd}
0 \arrow[r] & K'_{n+1} \arrow[r] \arrow[dr] & K_{n+1} \arrow[r] \arrow[dr] & K''_{n+1} \arrow[r] \arrow[dr] & 0 & \\
& 0 \arrow[r] & P'_{n+1} \arrow[r] \arrow[dl] \arrow[dd] & P_{n+1} \arrow[r] \arrow[dl] \arrow[dd] & P''_{n+1} \arrow[r] \arrow[dl] & 0 \\
0 \arrow[r] & K'_n \arrow[r] \arrow[dr] & K_n \arrow[r] \arrow[dr] & K''_n \arrow[r] \arrow[dr] & 0 & \\
& 0 \arrow[r] & P'_n \arrow[r] \arrow[d] & P_n \arrow[r] \arrow[d] & P''_n \arrow[r] \arrow[d] & 0 \\
& 0 \arrow[r] & P'_{n-1} \arrow[r] & P_{n-1} \arrow[r] & P''_{n-1} \arrow[r] & 0
\end{tikzcd}
$$

What happens to exact sequences under Ext?

Suppose that we have an exact sequence of modules.

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

Then there is an exact sequence

$$0 \to \hom(C,N) \to \hom(B,N) \to \hom(A,N) \to Ext^1_R(C,N) \to Ext^1_R(B,N) \to Ext^1_R(A,N) \to ...$$

Proof: Let $\mathcal{P}'$and $\mathcal{P}''$ be projective resolutions of A, C respectively.

By the Horseshoe lemma, we can construct a projective resolution $\mathcal{P}$ of A and chain maps to make the following diagram commute.

$$
\begin{tikzcd}
& {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & P'_1 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] & P''_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P'_0 \arrow[r] \arrow[d, "\epsilon'"'] & P_0 \arrow[r] \arrow[d, "\epsilon"'] & P''_0 \arrow[r] \arrow[d, "\epsilon''"'] & 0 \\
0 \arrow[r] & A \arrow[r, "i"'] & B \arrow[r, "p"'] & C \arrow[r] & 0
\end{tikzcd}
$$

Now take the deleted resolutions

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & P_2' \arrow[r] \arrow[d] & P_2 \arrow[r] \arrow[d] & P_2'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_1' \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] & P_1'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0' \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & P_0'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The rows still remain exact, since everything is projective we can take hom to obtain a new exact sequence of chain complex.

$$\begin{tikzcd}
 & {} & {} & {} & \\
0 & \hom(P_2', N) \arrow[l] \arrow[u] & \hom(P_2, N) \arrow[l] \arrow[u] & \hom(P_2'', N) \arrow[l] \arrow[u] & 0 \arrow[l] \\
0 & \hom(P_1', N) \arrow[l] \arrow[u] & \hom(P_1, N) \arrow[l] \arrow[u] & \hom(P_1'', N) \arrow[l] \arrow[u] & 0 \arrow[l] \\
0 & \hom(P_0', N) \arrow[l] \arrow[u] & \hom(P_0, N) \arrow[l] \arrow[u] & \hom(P_0'', N) \arrow[l] \arrow[u] & 0 \arrow[l] \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Now, take the long exact sequence to obtain a map

$$\to Ext_R^n(A,N) \to Ext_R^{n+1}(C,N) \to Ext_R^{n+1}\,(B,N) \to Ext_R^{n+1}(A,N) \to Ext_R^{n+2}(C,N) \to$$

If we start at the beginning we obtain

$$0 \to \hom(C,N) \to \hom(B,N) \to \hom(A,N) \to Ext_R^1(C,N) \to Ext_R^1(B,N) \to Ext_R^1(A,N) \to ...$$