Math 800 Commutative Algebra Notes: November 22

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1 Split implies Ext = 0

Definition 1.1. Let C, A be R-modules and

$$\xi : 0 \to A \to B \to C \to 0,$$

$$\xi' : 0 \to A \to B' \to C \to 0$$

be extensions of A by C. ξ and ξ' are *equivalent* if there exists $\phi : B \to B'$ such that

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "1_A"'] & B \arrow[r] \arrow[d, "\phi"'] & C \arrow[r] \arrow[d, "1_C"'] & 0 \\ 0 \arrow[r] & A \arrow[r] & B' \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

commutes.

Definition 1.2. Let $[\xi]$ denote the equivalence class of ξ under the above equivalence.

Let $e(C, A) = \{[\xi] : \xi \text{ is an extension from } A \text{ to } C\}$.

Our goal is to prove we get a set bijection:

$$\psi : e(C, A) \to \operatorname{Ext}^1(C, A).$$

Given an extension

$$\xi : 0 \to A \to B \to C \to 0$$

and a projective resolution

$$\to P_2 \to P_1 \to P_0 \to C \to 0$$

we form the following diagram:

$$\begin{tikzcd} \arrow[r] & P_2 \arrow[r, "d_2"] \arrow[d] & P_1 \arrow[r, "d_1"] \arrow[d, "\alpha"'] & P_0 \arrow[r, "d_0"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "1_C"'] & 0 \\ & 0 \arrow[r] & A \arrow[r, "i"'] & B \arrow[r, "p"'] & C \arrow[r] & 0 \end{tikzcd}$$

Figure 1

Since P_0 is projective we have

$$\begin{tikzcd} & P_0 \arrow[dl, "\beta"'] \arrow[d, "d_0"] & \\ B \arrow[r, "p"'] & C \arrow[r] & 0 \end{tikzcd}$$

giving β in Figure 1.

We also have $\beta d_1 : P_1 \to B$. Since $\operatorname{im} i = \ker p$ and $p\beta d_1 = d_0 d_1 = 0$, we have $\operatorname{im} \beta d_1 \subseteq \ker p = \operatorname{im} p$.

We also have

$$\begin{tikzcd} & P_1 \arrow[dl, "\alpha"'] \arrow[d, "\beta d_1"] & \\ A \arrow[r, "i"'] & \operatorname{im} i \arrow[r] & 0 \end{tikzcd}$$

by projectivity of P_1 giving α in Figure 1. Likewise $\operatorname{im} \alpha \subseteq \ker i = 0$ and thus Figure 1 is commutative.

Notice $\alpha d_2 = 0$ so with

$$d_2^* : \operatorname{Hom}(P_2, A) \to \operatorname{Hom}(P_1, A)$$

we have $d_2^* \alpha = \alpha_2 = 0$.

Thus α is a cocycle (in the Hom groups used to build Ext).

Furthermore any two fillings are homotopic. Suppose we have:

$$\begin{tikzcd} \arrow[r] & P_2 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d, "\alpha"'] & P_0 \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "1_C"'] & 0 \\ & 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

and

$$\begin{tikzcd} \arrow[r] & P_2 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d, "\alpha'"'] & P_0 \arrow[r] \arrow[d, "\beta'"'] & C \arrow[r] \arrow[d, "1_C"'] & 0 \\ & 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

Consider

$$\begin{tikzcd} \arrow[r] & P_2 \arrow[r, "d_2"] \arrow[d] & P_1 \arrow[r, "d_1"] \arrow[dl, "s_1"'] \arrow[d, "\alpha-\alpha'"'] & P_0 \arrow[r, "d_0"] \arrow[dl, "s_0"'] \arrow[d, "\beta-\beta'"'] & C \arrow[r] \arrow[dl, "s_{\text{-}1}"'] \arrow[d, "0_C"'] & 0 \\ & 0 \arrow[r] & A \arrow[r, "i"'] & B \arrow[r, "p"'] & C \arrow[r] & 0 \end{tikzcd}$$

We want:

$$\begin{aligned}
\beta - \beta' &= is_0 + s_{\text{-}1}d_0 \\
\alpha - \alpha' &= 0s_1 + s_0 d_0 \\
0 &= ps_{\text{-}1} + 1_C \\
0 &= 0 + s_1 d_2.
\end{aligned}$$

Take $s_{\text{-}1} = s_1 = 0$ then we just need

$$\begin{aligned}
\beta - \beta' &= is_0 \\
\alpha - \alpha' &= s_0 d_0.
\end{aligned}$$

We know that $p(\beta - \beta') = 0d_0 = 0$ so given $p_0 \in P_0$, $(\beta - \beta')(p_0) \in \ker p = \operatorname{im} i$. Since i is injective there exists a unique $a \in A$ such that $i(a) = (\beta - \beta')(p_0)$. Let $s_0 : p_0 \mapsto a$ and then $\beta - \beta' = is_0$. Also for $p \in P_1$ $i(\alpha - \alpha')(p_1) = (\beta - \beta')d_1(p_1) = is_0 d_1(p_1)$ and since i is injective $\alpha - \alpha' = s_0 d_1$.

This property is true in general and is called the Comparison Theorem (Rotman Theorem 10.46).

Since $\alpha - \alpha' \in \operatorname{im} d_1^*$, ($d_1^* : \operatorname{Hom}(P_1, A) \to \operatorname{Hom}(P_0, A)$), $\psi := \alpha + \operatorname{im} d_1^* \in \operatorname{Ext}^1(C, A)$ is a well-defined map.

Given an extension ξ, we need to check that ψ does not depend on the choice of element in $[\xi]$. Take the diagram:

$$\begin{tikzcd}
{} \arrow[r] & P_2 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d, "\alpha''"] \arrow[dd, bend right, "\alpha"'] & P_0 \arrow[r] \arrow[d, "\beta'"] \arrow[dd, bend right, "\beta"'] & C \arrow[r] \arrow[d, "1_C"] & 0 \\
& 0 \arrow[r] & A \arrow[r, "i'"'] \arrow[d, "1_A"] & B' \arrow[r, "p'"'] \arrow[d, "\phi"] & C \arrow[r] \arrow[d, "1_C"] & 0 \\
& & A \arrow[r, "i"'] & B \arrow[r, "p"'] & C \arrow[r] & 0
\end{tikzcd}$$

Consider $\alpha - \alpha''$. Take $p_1 \in P - 1$,

$$\begin{aligned}
i(\alpha - \alpha'')(p_1) &= i\alpha(p_1) - \phi i'\alpha(p_1) \\
&= \beta d{\cdot}1(p_1) - \phi\beta' d_1(p_1) \\
&= \beta(d_1(p) - d_1(p)) \\
&= 0.
\end{aligned}$$

So $\alpha - \alpha'' = 0$ and any other α'', β' for

$$0 \to A \to B \to C \to 0$$

will be homotopic.

Define

$$\begin{array}{rccc}
\psi : & e(C, A) & \to & \operatorname{Ext}^1(C, A) \\
& [\xi] & \mapsto & \alpha + \operatorname{im} d_1^*
\end{array},$$

which is well defined.

Lemma 1.3. *Let*

$$\Xi : 0 \to X_1 \to X_0 \to C \to 0$$

be an extension of X_1 by C.

Given $\alpha : X_1 \to A$ consider:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_1 & \xrightarrow{j} & X_0 & \xrightarrow{\epsilon} & C & \longrightarrow & 0 \\
 & & {\scriptstyle\alpha}\downarrow & & & & {\scriptstyle 1_C}\downarrow & & \\
 & & A & & & & C & &
\end{array}$$

Then

1. *the diagram can be completed to:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_1 & \xrightarrow{j} & X_0 & \xrightarrow{\epsilon} & C & \longrightarrow & 0 \\
 & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle 1_C}\downarrow & & \\
 & & A & \xrightarrow[i]{} & B & \xrightarrow[\eta]{} & C & &
\end{array}$$

2. *and any two bottom rows of such completions are equivalent.*

Proof. 1. Let $S = \{(j(x), -\alpha(x)) \in X_0 \oplus A : x \in X_1\}$. S is a submodule of $X_0 \oplus A$. Let $B = {X_0 \oplus A}/S$ then

$$\begin{array}{ccc}
X_1 & \xrightarrow{\quad j \quad} & X_0 \\
{\scriptstyle\alpha}\Big\downarrow & & \Big\downarrow{\scriptstyle \phi : x \mapsto (x,0)} \\
A & \xrightarrow[i : a \mapsto (0,a)]{} & B
\end{array}$$

commutes by definition. B is called the pushout, it is dual to the pullback and satisfies analogous universal properties.

Define

$$\begin{array}{rccc}
\eta : & B & \to & C \\
 & (x_0, a) & \mapsto & \epsilon(x_0),
\end{array}$$

then the diagram commutes by construction.

Lastly we check exactness at B. For $(x, a) \in \ker \eta$:

$$\begin{aligned}
(x_0, a) \in \ker \eta &\iff \epsilon(x_0) = 0 \\
&\iff x_0 \in \ker \epsilon \\
&\iff x_0 \in \operatorname{im} j \\
&\iff \exists x_1 \in X_1 \text{ s.t } j(x_1) = x_0 \\
&\iff \exists x_1 \in X_1 \text{ s.t } j(x_1) = x_0 \text{ and } \exists a'1 \in S \text{ s.t. } \alpha(x_1) = a' - a \\
&\iff \exists a' \in S \text{ s.t } (x_0, a - a') \in S \\
&\iff \exists a' \in S \text{ s.t } (x_0, a) + S = (x_0, a') + S \\
&\iff (x_0, a) \in \operatorname{im} i.
\end{aligned}$$

2. See Rotman Lemma 10.87 (ii).

□

Definition 1.4. Given Ξ and α as above, let $\alpha\Xi$ denote the equivalence class of the extension above.
Proposition 1.5. *The function $\psi : e(C, A) \to \operatorname{Ext}^1(C, A)$ is a bijection.*

Proof. Chose a projective resolution of C:

$$P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to C \to 0$$

and a 1 cocycle $\alpha : P_1 \to A$.

Let

$$\Xi : 0 \to {}^{P_1}/_{\operatorname{im} d_2} \to P_0 \to C \to 0.$$

Since $\alpha d_2 = 0$, $\alpha(\operatorname{im} d_2) = 0$ and so α induces

$$\alpha' : {}^{P}/_{\operatorname{im} s_2} \to A.$$

We have

$$\begin{tikzcd}
\Xi : 0 \arrow[r] & {}^{P_1}/_{\operatorname{im} d_2} \arrow[r] \arrow[d, "\alpha'"'] & P_0 \arrow[r] & C \arrow[r] \arrow[d] & 0 \\
& A & & C &
\end{tikzcd}$$

So by the lemma we get:

$$\begin{tikzcd}
\Xi : 0 \arrow[r] & {}^{P_1}/_{\operatorname{im} d_2} \arrow[r] \arrow[d, "\alpha'"'] & P_0 \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d] & 0 \\
\alpha'\Xi : 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

Define

$$\begin{array}{rccc}
\theta : & \operatorname{Ext}^1(C, A) & \to & e(C, A) \\
& \alpha + \operatorname{im} d^* & \mapsto & [\alpha'\Xi] \,.
\end{array}$$

First we check θ is well defined. Suppose $\zeta : P_1 \to A$ is another cocycle. α and ζ are homotopic so there exists $s : P_0 \to A$ such that $\zeta - \alpha = sd_1$. Then,

$$\begin{tikzcd}
P_2 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d, "\alpha+sd_1"'] & P_0 \arrow[r] \arrow[d, "\beta+is"'] & C \arrow[r] \arrow[d, "1_C"'] & 0 \\
0 \arrow[r] & A \arrow[r, "i"'] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
P_2 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d, "\alpha"'] & P_0 \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "1_C"'] & 0 \\
0 \arrow[r] & A \arrow[r, "i"'] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

have the same bottom row and so $[\alpha'\Xi] = [\zeta'\Xi]$.

Next we check that $\psi\theta = 1$. Take $\alpha + \operatorname{im} d_1^* \in \operatorname{Ext}^1(C, A)$. We have:

$$\begin{tikzcd}
0 \arrow[r] & {}^{P_1}/_{\operatorname{im} d_2} \arrow[r] \arrow[d, "\alpha'"'] & P_0 \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "1_C"'] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

with the bottom row in $\theta(\alpha + \operatorname{im} d_1^*)$. This diagram implies:

$$\begin{tikzcd}
P_2 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d, "\alpha"'] & P_0 \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "1_C"'] & 0 \\
0 \arrow[r] & A \arrow[r, "i"'] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

and thus $\alpha \in \psi\theta(\alpha + \operatorname{im} d_1^*)$.

Finally we check $\theta\psi = 1$. This is the same as above in reverse using the lemma to show that the original and new extensions are in the same class. $\square$

Theorem 1.6. *Let A, C be R-modules.*

Every extension of A by C splits if and only if $\operatorname{Ext}^1(C, A) = 0$.

Proof. If every extension splits then $|e(C, A)| = 1$ so $\operatorname{Ext}^1(C, A) = 0$.

If $\operatorname{Ext}^1(C, A) = 0$ then $|e(C, A) = 1$. The set of split extensions are an equivalence class of $e(C, A)$ and so the are the only thing in $e(C, A)$. $\square$

Note: We have more than this theorem because we saw that Ext^1 counts extensions in some sense.
Example 1.7. We saw previously that for B, an abelian group, $\operatorname{Ext}^1_{\mathbb{Z}}({}^{\mathbb{Z}}/_{n\mathbb{Z}}, B) \cong {}^{B}/_{nB}$.

Take p, prime then $\operatorname{Ext}^1_{\mathbb{Z}}({}^{\mathbb{Z}}/_{p\mathbb{Z}}, {}^{\mathbb{Z}}/_{p\mathbb{Z}}) \cong {}^{\mathbb{Z}}/_{p\mathbb{Z}}$. So there are p equivalence classes of extensions:

$$0 \to {}^{\mathbb{Z}}/_{p\mathbb{Z}} \to A \to {}^{\mathbb{Z}}/_{p\mathbb{Z}} \to 0.$$

Since $|A| = p^2$ is an abelian group. $A \cong {}^{\mathbb{Z}}/_{p\mathbb{Z}} \oplus {}^{\mathbb{Z}}/_{p\mathbb{Z}}$ and the extension splits or $A \cong ({}^{\mathbb{Z}}/_{p^2\mathbb{Z}})$. Therefore there must be $p-1$ ways to put ${}^{\mathbb{Z}}/_{p\mathbb{Z}}$ into ${}^{\mathbb{Z}}/_{p^2\mathbb{Z}}$ injective. This is true as they are $p-1$ nonzero equivalence classes modulo p.

Version 1.0.

References

[1] Joseph Rotman, *Advanced Modern Algebra*, Graduate Studies in Mathematics (2002), Prentice Hall.