MATH 817 ASSIGNMENT 5

DUE NOVEMBER 19, 2009, IN CLASS

If your assignment must be late for any reason please notify me (by email, phone or in person) **before** the assignment is due. There will be no retroactive lates.

(1) (Isaacs problem 11.8) Let G be an X-group with the property that whenever $\mathcal{H}$ is a linearly ordered collection of proper X-subgroups of G, then $\bigcup \mathcal{H} < G$. Show that G is finitely generated. *Hint: Zorn's lemma.*

(2) (Isaacs problem 12.24) Let B and C be subrings of A with $1_B = 1_C = 1_A$. Write A_B for A viewed as a right B module and ${}_CA$ for A viewed as a left C module. Let

$$R = \left\{ \begin{bmatrix} b & 0 \\ a & c \end{bmatrix} : a \in A, b \in B, c \in C \right\} \text{ and } I = \left\{ \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix} : a \in A \right\}$$

(a) Show that R is a ring and that I is an ideal of R.
(b) Show that $R/I \cong B \oplus C$.
(c) Show that I is noetherian or artinian as a right R module iff A_B is noetherian or artinian.
(d) Show that I is noetherian or artinian as a left R module iff ${}_CA$ is noetherian or artinian.
(e) In the case $A = B = \mathbb{R}$ and $C = \mathbb{Q}$, show that R is right noetherian and right artinian but is neither left noetherian nor left artinian.

(3) Let V and W be finite dimensional vector spaces over a field F. Let $S : V \to V$ be a linear transformation with matrix A and let $T : W \to W$ be a linear transformation with matrix B. What is the matrix of $S \otimes T : V \otimes_F W \to V \otimes_F W$?

(4) Let $C = \mathbb{Z}/2\mathbb{Z}$. Let $C(x)$ be the field of rational functions in x over C. Prove that $C(x) \otimes_{C(x^2)} C(x)$ has nonzero nilpotent elements.

(5) (Isaacs problem 13.7) Fix a prime p and consider the ring

$$L_p = \{m/n \in \mathbb{Q} : m, n \in Z, p \not| n\}.$$

Compute $J(L_p)$.