MATH 817 ASSIGNMENT 5 SOLUTIONS

(1) Following Isaacs' hint let $\mathcal{S}$ be the set of all X-subgroups $H \subseteq G$ for which there is no finite subset $Y \subseteq G$ with $G = \langle H \cup Y \rangle$.

Suppose $\mathcal{S}$ had a maximal element H. Then $H < G$ since otherwise the empty set would contradict the condition to be in $\mathcal{S}$. Take any $y \in G \setminus H$. Let $S = \langle H \cup \{y\} \rangle$. Suppose Y were finite such that $G = \langle S \cup Y \rangle$. Then $Y \cup \{y\}$ would contradict the condiditon of $\mathcal{S}$ for H so no such Y can exist. Thus $H < S \in \mathcal{S}$ contradicting the maximality of H.

Let $\mathcal{H}$ be any nonempty linearly ordered subset of $\mathcal{S}$. Let $K = \bigcup \mathcal{H}$. By assumption $K < G$. Suppose $G = \langle K \cup Y \rangle$ for some finite set Y. Then $G = \bigcup_{H \in \mathcal{H}} H \cup Y$ and each $H \cup Y < G$ since $H \in \mathcal{S}$ which is a contradiction. Thus $K \in \mathcal{S}$.

Suppose G is not finitely generated. Then $\mathcal{S} \neq \varnothing$ since $0 \in \mathcal{S}$. So we can apply Zorn's lemma to get a maximal element of $\mathcal{S}$ which is a contradiction. Thus G is finitely generated.

(2) (a) It is clear that R is a group under addition with additive identity the zero matrix. It contains the identity matrix which is the multiplicative identity. It remains only to show that it is closed under matrix multiplication. Compute

$$\begin{bmatrix} b & 0 \\ a & c \end{bmatrix} \begin{bmatrix} b' & 0 \\ a' & c' \end{bmatrix} = \begin{bmatrix} bb' & 0 \\ ab' + ca' & cc' \end{bmatrix} \in R$$

I is also clearly a group under addition. It remaisn to show that it is closed under right and left multiplication from R. Compute

$$\begin{bmatrix} b & 0 \\ a & c \end{bmatrix} \begin{bmatrix} 0 & 0 \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ ci & 0 \end{bmatrix} \in I$$

$$\begin{bmatrix} 0 & 0 \\ i & 0 \end{bmatrix} \begin{bmatrix} b & 0 \\ a & c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ ia & 0 \end{bmatrix} \in I$$

(b) Let $\phi : R/I \to B \oplus C$ be defined by

$$\phi \left(I + \begin{bmatrix} b & 0 \\ a & c \end{bmatrix} \right) = (b, c)$$

ϕ is well defined as if $I + \begin{bmatrix} b & 0 \\ a & c \end{bmatrix} = I + \begin{bmatrix} b' & 0 \\ a' & c' \end{bmatrix}$ then $\begin{bmatrix} b - b' & 0 \\ a - a' & c - c' \end{bmatrix} \in I$ so $b = b'$, $c = c'$. ϕ is onto by the definition of R. Finally ϕ is one-to-one as if $\phi \left(I + \begin{bmatrix} b & 0 \\ a & c \end{bmatrix} \right) = \phi \left(I + \begin{bmatrix} b' & 0 \\ a' & c' \end{bmatrix} \right)$ then $b = b'$ and $c = c'$ so $\begin{bmatrix} b & 0 \\ a & c \end{bmatrix} - \begin{bmatrix} b' & 0 \\ a' & c' \end{bmatrix} \in I$.

(c) Note that the right action of R on I (calculated above) consists of right multiplication of B on A in the lower left corner. Thus I as a right R module has the same structure as A_B. Thus one is artinian or noetherian iff the other is.

(d) Note that the left action of R on I (calculated above) consists of left multiplication of C on A in the lower left corner. Thus I as a right R module has the same structure as ${}_CA$. Thus one is artinian or noetherian iff the other is.

(e) $\mathbb{R}$ is not artinian as a module over $\mathbb{Q}$ because we have the chain of ideals

$$\langle \pi \rangle > \langle \pi^2 \rangle > \cdots$$

since π is transcendental, and $\mathbb{R}$ is not noetherian as a module over $\mathbb{Q}$ because, by cardinality, we can choose a sequence of algebraically independent transcendentals $(z_1, z_2, \cdots)$ and so we have the chain of ideals

$$\langle z_1 \rangle < \langle z_1, z_2 \rangle < \cdots$$

So by part 2d I is neither artinian nor noetherian as a left R module, but I is an ideal of R so the same is true for R.

One the other hand $\mathbb{R}$ is artinian and noetherian as a module over itself because it is a field. So by part 2c I is both artinian and noetherian as a right R module. Also by part 2b $R/I \cong \mathbb{R} \oplus \mathbb{Q}$. R/I is a right R-module by right multiplication by R and the action on $\mathbb{R} \oplus \mathbb{Q}$ is by right multiplication by $\mathbb{R}$ in the first coordinate and right multiplication by $\mathbb{Q}$ in the second coordinate. So R/I as a right R-module is the field $\mathbb{R} \oplus \mathbb{Q}$ with coordinatewise multiplication acting on itself which thus is both noetherian and artinian. Hence so is R, as required.

(3) Let $v_1, v_2, \ldots, v_n$ be a basis for V and $w_1, w_2, \ldots, w_m$ be a basis for W. Let $A = (a_{i,j})$, $B = (b_{i,j})$. Then $V \otimes W(v_i \otimes w_j) = V(v_i) \otimes W(w_j) = \sum_{k,\ell} a_{i,k} b_{j,\ell} v_k \otimes w_\ell$. So using the ordered basis $v_1 \otimes w_1, v_1 \otimes w_2, \ldots, v_2 \otimes w_1, v_2 \otimes w_2, \ldots, v_n \otimes w_m$ the matrix of $V \otimes W$ is, in block form

$$\begin{bmatrix} a_{1,1}B & a_{1,2}B & \cdots & a_{1,n}B \\ a_{2,1}B & a_{2,2}B & \cdots & a_{2,n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1}B & a_{n,2}B & \cdots & a_{n,n}B \end{bmatrix}$$

which is known as the Kronecker product of A and B.

(4) $x \otimes 1 + 1 \otimes x \neq 0$ in $C(x) \otimes_{C(x^2)} C(x)$ since x and 1 in $C(x)$ are linearly independent over $C(x^2)$. But

$$\begin{aligned} (x \otimes 1 + 1 \otimes x) &= x^2 \otimes 1 + x \otimes x + x \otimes x + 1 \otimes x^2 \\ &= x^2(1 \otimes 1) + 0 + x^2(1 \otimes 1) \\ &= 0 \end{aligned}$$

(5) An element of L_p is not a unit precisely if p divides the numerator of the fraction in lowest form. Let I be the set of all such elements. Then $I = pL_p$ is an ideal. Thus it is the only maximal ideal. Thus $J(L_p) = I$.