HOMEWORK 2 SOLUTIONS

MATH 818, FALL 2010

Sh, I.4.9: We have

$$f : \mathbb{P}^2 \to \mathbb{P}^2$$
$$(x_1 : x_2 : x_3) \mapsto (x_1x_2 : x_0x_2 : x_0x_1)$$

This is a rational map, and it is regular unless $x_1x_2 = 0$, $x_0x_2 = 0$, $x_0x_1 = 0$. That is f is regular except at $(0 : 0 : 1)$, $(0 : 1 : 0)$, and $(1 : 0 : 0)$. f is its own inverse and hence f is a birational map. f is an isomorphism on $\mathbb{P}^2 \backslash V(x_1) \cup V(x_2) \cup V(x_3)$ since if no coordinate is 0 then no coordinate is zero after applying f.

Sh, I.5.7: $k[\mathbb{A}^1] = k[t]$. $k[V(y^2 = x^3)] = k[x, y]/\langle y^2 = x^3\rangle$, and $f^* : k[x, y]/\langle y^2 = x^3\rangle \to k[t]$ by $f(p(x, y)) = p(t^2, t^3)$. So the question is asking if $k[t]$ is integral over $k[t^2, t^3]$. This is the case because we need only to check that t is integral over $k[t^2, t^3]$ which it is since it satisfies the monic polynomial $T^2 = t^2$.

Sh, I.5.8: Embed $\mathbb{A}^r$ in $\mathbb{P}^r$. Let $\mathbb{P}^{r-1}_\infty$ be the points at infinity. Let $E = \overline{L} \cap \mathbb{P}^r$. Then projection parallel to L in $\mathbb{A}^r$ is the same as projecting away from E in $\mathbb{P}^r$ (This is because lines paralell to L in $\mathbb{A}^r$ are precisely lines in $\mathbb{P}^r$ which go through E.). Thus from Theorem I.5.7 of Shafarevich we know that ϕ_L is finite when $E \notin \overline{X}$. So

$$S \subset \overline{X} \cap \mathbb{P}^{r-1}_\infty.$$

On the other hand suppose $E \in \overline{X}$. Let $t_0, \dots, t_r$ be projective coordinates on $\mathbb{P}^r$ with $\mathbb{P}^{r-1}_\infty$ being $t_0 = 0$. Use a projective automorphism (linear even!) to make $L = V(t_1, \dots, t_{r-1})$ and the $(r-1)$-dimensional subspace Y not containing L equal $V(t_r)$. Then $E = (0 : 0 : \cdots : 1)$ and

$$\phi_L : X \to Y$$
$$(t_1, \dots, t_r) \mapsto (t_1, \dots, t_{r-1})$$

Suppose ϕ_L were finite. Consider t_r as a function on X; then t_r satisfies an equation

$$t_r^k + a_{k-1}t_r^{k-1} + \cdots + a_0 = 0$$

in $k[Y]$. But $E \in \overline{X}$ means t_r tends to infinity on X. So consider $y \in Y$, $x \in \phi_L^{-1}(y)$; we have

$$t_r(x)^k + a_{k-1}(y)t_r(x)^{k-1} + \cdots + a_0(y) = 0$$

Choosing x and y to let t_r tend to infinity we get a contradiction, so ϕ_L is not finite.

Finally if $r = 2$ and $X = V(xy = 1)$ then $\overline{X} = V(xy = z^2)$, and so $S = \overline{X} \cap \mathbb{P}^1_\infty = \{(1 : 1 : 0)\}$.

F, 3-2: Assume the characteristic of k is 0.

(a) Solve

$$Y^3 - Y^2 + X^3 - X^2 + 3Y^2X + 3X^2Y + 2XY = 0$$
$$3Y^2 - 2Y + 6YX + 3X^2 + 2X = 0$$
$$3X^2 - 2X + 3Y^2 + 6XY + 2Y = 0$$

Adding $0 = 3eq_1 - Yeq_2 - Xeq_3 = -X^2 - Y^2 + 2XY$ so $X + Y = 0$. Subbing $X = -Y$ into the second equation gives $Y = 0$, and $(0,0)$ satisfies all the equations, so the only singular point is $(0,0)$.
Taking the lowest degree part we have $-X^2 - Y^2 + 2XY = 0$ so $(0,0)$ is a double point and the only tangent line is $Y = X$.

(b) Solve

$$Y^4 + X^4 - X^2Y^2 = 0$$
$$4Y^3 - 2X^2Y = 0$$
$$4X^3 - 2XY^2 = 0$$

The second equation gives $Y = 0$ or $2Y^2 = X^2$. The third equation gives $X = 0$ or $2X^2 = Y^2$. Thus $Y = 0 \Leftrightarrow X = 0$ and $(0,0)$ is a solution to the system. The other possibility is $2Y^2 = X^2$ and $2X^2 = Y^2$ but these cannot be simultaneously satisfied unless again $X = Y = 0$. Thus the only singular point is $(0,0)$.
This polynomial is homogeneous and its linear factors are $2X \pm Y(i \pm \sqrt{3})$ thus the point is a quadruple point and has those four tangent lines.

(c) Solve

$$Y^3 + X^3 - 3X^2 - 3Y^2 + 3XY + 1 = 0$$
$$3Y^2 - 6Y + 3X = 0$$
$$3X^2 - 6X + 3Y = 0$$

Adding $0 = 3eq_1 + (1-Y)eq_2 + (1-X)eq_3 = (X-1)(Y-1)$. So $X = 1$ or $Y = 1$. Subbing $X = 1$ into equation 3 gives $Y = 1$, and similarly starting with $Y = 1$, so we have one singular point $(1,1)$.
Translating the singular point to the origin

$$(Y+1)^3 + (X+1)^3 - 3(X+1)^2 - 3(Y+1)^2 + 3(X+1)(Y+1) + 1 = X^3 + Y^3 + 3XY$$

The lowest degree part is $3XY$ and so the point is a double point and the tangents are (translating back) $X = 1$ and $Y = 1$.

(d) Solve

$$Y^2 + (X^2 - 5)(2X^2 - 5)^2 = 0$$
$$2Y = 0$$
$$2X(6X^2 - 25)(2X^2 - 5) = 0$$

So from equation 2 $Y = 0$, and the only common roots of equation 2 and equation 1 with $Y = 0$ are $2X^2 = 5$, that is $X = \pm\sqrt{5/2}$. So the singular points are $(\sqrt{5/2}, 0)$ and $(-\sqrt{5/2}, 0)$.

First take $(\sqrt{5/2}, 0)$. Translating to the origin we get

$$Y^2 + ((X + \sqrt{5/2})^2 - 5)(2(X + \sqrt{5/2})^2 - 5)^2$$

Fortunately we only need the lowest degree part: $Y^2 - 100X^2$, and the lowest degree part will be the same (translated) for the other singular point. Thus both are double points with tangents (translated back) the appropriate two of $Y = \pm 10(X \pm \sqrt{5/2})$.

F, 3-8: (a) Translate P and Q to the origin. Composing T with these two translations we still get a polynomial map. Then $F^T(x, y) = F(T_1(x, y), T_2(x, y))$. But T takes the origin to the origin, so T_1 and T_2 both have no constant terms. Thus the lowest degree part of F^T has degree at least the lowest degree part of F.

(b) Continuing with the notation and assumtions above, write

$$F = f_m + f_{m+1} + \cdots + f_n$$

with f_i homogeneous of degree i. As above applying T cannot decrease degrees, so the lowest degree part of F^T is the lowest degree part of f_m^T. Thus only the degree 1 part of F plays a role in the multiplicities. Suppose the Jacobian is invertible at the origin. Then the degree 1 part of F is invertible with a polynomial inverse L (of degree 1). And so $m_P(F) \leq m_Q(F^T) \leq m_P((F^T)^L) = m_P(F)$. Thus $m_Q(F^T) = m_P(F)$.

(c) Using the example Fulton gives, $m_P(F) = 1$ and $F^T = Y - X^4$ so $m_P(F^T) = 1$. However the Jacobian is

$$\begin{bmatrix} 2X & 0 \\ 0 & 1 \end{bmatrix}$$

which is not invertible at the origin.

F, 3-13: Translate P to the origin and take $0 \leq n < m_P(F)$. Then $\mathfrak{m} = \langle X, Y \rangle$. Thus $\mathfrak{m}^{n+1}/\mathfrak{m}^n$ is the vector space in $k[F]$ generated by homogeneous polynomials of degree n. But $n < m_P(F)$, so F has no terms of degree less than or equal to n. Thus F introduces no relations on $\mathfrak{m}^{n+1}/\mathfrak{m}^n$. Therefore $\dim \mathfrak{m}^{n+1}/\mathfrak{m}^n = n + 1$.

If P is a simple point then $\dim \mathfrak{m}/\mathfrak{m}^2 = 1$ as n is sufficiently large for the theorem (Theorem 2 in Fulton) to apply. If P is not a simple point then $\dim \mathfrak{m}/\mathfrak{m}^2 = 2$ by the above argument.

F, 3-20: P is a simple point on F so we are trying to show

$$\mathrm{ord}_P^F(G + H) \geq \min\{\mathrm{ord}_P^F(G), \mathrm{ord}_P^F(H)\}.$$

But this is the ultrametric triangle inequality which is satisfied by ord. (For a proof, suppose t is the uniformizer, and say t divides G exactly n times and H exactly m times. Then certainly t divides $G + H$ $\min\{m, n\}$ times, and perhaps more if there is cancellation.)

This does not hold if P is not a simple point on F because we have the following example. Let $P = (0, 0)$ and $F = Y^2 - X^2(X + 1)$. Let $G = X + Y$ and $H = X - Y$. Then

$$\mathrm{In}(P, F \cap G) = \mathrm{In}(P, V((X+Y)(X-Y) - X^3) \cap V(X+Y)) = \mathrm{In}(P, V(X^3) \cap V(X+Y)) = 3.$$

Similarly

$$\mathrm{In}(P, F \cap H) = \mathrm{In}(P, V((X+Y)(X-Y) - X^3) \cap V(X-Y)) = \mathrm{In}(P, V(X^3) \cap V(X-Y)) = 3.$$

But $G+H=2X$ and

$$\mathrm{In}(P, F\cap V(2X)) = \mathrm{In}(P, V(Y^2)\cap V(2X)) = 2.$$

which does not satisfy the inequality.

F, 5-3: (a) We have

$$Y^2Z - X(X-2Z)(X+Z) = 0$$
$$Y^2 + X^2 - 2XZ = 0$$

Solving the second for Y^2 and subbing into the first we get

$$Y^2(X+2Z) = 0$$

So $Y=0$ or $X=-2Z$. If $Y=0$ then we have

$$X(X-2Z)(X+Z) = 0 \text{ and } X(X-2Z) = 0$$

so we have the two points $(0:0:1)$ and $(2:0:1)$. If $X=-2Z$ then we have

$$Y^2X - 8Z^3 = 0 \text{ and } Y^2 + 8Z^2 = 0$$

so we have the two points $(2:2\sqrt{2}:-1)$ and $(2:-2\sqrt{2}:-1)$. Calculate the intersection multiplicities:

$(0:0:1)$: Dehomogenize with $Z=1$. Let $P=(0,0)$, calculate

$$\begin{aligned}
&\mathrm{In}(P, V(Y^2 - X(X-2)(X+1))\cap V(Y^2+X(X-2))) \\
&= \mathrm{In}(P, V(X(X-2)(X+2))\cap V(Y^2+X(X-2))) \\
&= \mathrm{In}(P, V(X)\cap V(Y^2+X(X-2))) + \mathrm{In}(P, V(X-2)\cap V(Y^2+X(X-2))) \\
&\quad + \mathrm{In}(P, V(X+2)\cap V(Y^2+X(X-2))) \\
&= \mathrm{In}(P, V(X)\cap V(Y^2)) + 0 + 0 \\
&= 2
\end{aligned}$$

$(2:0:1)$: Dehomogenize with $Z=1$. Let $P=(2,0)$.

$$\begin{aligned}
&\mathrm{In}(P, V(Y^2 - X(X-2)(X+1))\cap V(Y^2+X(X-2))) \\
&= \mathrm{In}(P, V(X(X-2)(X+2))\cap V(Y^2+X(X-2))) \\
&= 0 + \mathrm{In}(P, V(X-2)\cap V(Y^2+X(X-2))) + 0 \\
&= \mathrm{In}(P, V(X-2)\cap V(Y^2)) \\
&= 2
\end{aligned}$$

$(2:2\sqrt{2}:-1)$: Dehomogenize with $Z=1$. Let $P(-2:-2\sqrt{2})$.

$$\begin{aligned}
&\mathrm{In}(P, V(Y^2 - X(X-2)(X+1))\cap V(Y^2+X(X-2))) \\
&= \mathrm{In}(P, V(X(X-2)(X+2))\cap V(Y^2+X(X-2))) \\
&= 0 + 0 + \mathrm{In}(P, V(X+2)\cap V(Y^2+X(X-2))) \\
&= \mathrm{In}(P, V(X+2)\cap V(Y^2+8)) \\
&= \mathrm{In}(P, V(X+2)\cap V(Y+2\sqrt{2})) + \mathrm{In}(P, V(X+2)\cap V(Y-2\sqrt{2})) \\
&= 1
\end{aligned}$$

$(2:-2\sqrt{2}:-1)$: By the same calculation as the previous point but with the last two terms switched we get again an intersection multiplicity of 1.

(b) We have

$$(X^2+Y^2)Z + X^3 + Y^3 = 0$$
$$X^3 + Y^3 - 2XYZ = 0$$

Subbing the second into the first we get $Z(X^2+Y^2+2XY) = 0$ so $Z = 0$ or $X+Y=0$. If $Z=0$ we have $X^3+Y^3=0$ so we get the points $(1:-1:0)$, $(1:-e^{2\pi i/3}:0)$, $(1:-e^{4\pi i/3}:0)$. If $X-Y=0$ then we get $2Y^2Z=0$ so we get the new point $(0:0:1)$. Calculate the intersection multiplicities. For the first three cases dehomogenize with $X=1$. That is, calculate

$$\begin{aligned}
&\mathrm{In}(P, V((Y^2+1)Z+Y^3+1) \cap V(Y^3+1-2YZ)) \\
&= \mathrm{In}(P, V((Y^2+2Y+1)Z) \cap V(Y^3+1-2YZ)) \\
&= \mathrm{In}(P, V(Z) \cap V(Y^3+1-2YZ)) + 2\mathrm{In}(P, V(Y+1) \cap V(Y^3+1-2YZ)) \\
&= \mathrm{In}(P, V(Z) \cap V(Y^3+1)) + 2\mathrm{In}(P, V(Y+1) \cap V(2Z)) \\
&= \mathrm{In}(P, V(Z) \cap V(Y+1)) + \mathrm{In}(P, V(Z) \cap V(Y+e^{2\pi i/3})) \\
&\quad + \mathrm{In}(P, V(Z) \cap V(Y+e^{4\pi i/3})) + 2\mathrm{In}(P, V(Y+1) \cap V(2Z))
\end{aligned}$$

$(1:-1:0)$: Let $P=(-1,0)$, continuing the above calculation

$$\mathrm{In}(P, V((Y^2+1)Z+Y^3+1) \cap V(Y^3+1-2YZ)) = 1+0+0+2 = 3$$

$(1:-e^{2\pi i/3}:0)$: Let $P=(-e^{2\pi i/3},0)$.

$$\mathrm{In}(P, V((Y^2+1)Z+Y^3+1) \cap V(Y^3+1-2YZ)) = 0+1+0+0 = 1$$

$(1:-e^{4\pi i/3}:0)$: Let $P=(-e^{4\pi i/3},0)$.

$$\mathrm{In}(P, V((Y^2+1)Z+Y^3+1) \cap V(Y^3+1-2YZ)) = 0+0+1+0 = 1$$

$(0:0:1)$: This time dehomogenize with $Z=1$. Let $P=(0,0)$.

$$\mathrm{In}(P, V((X^2+Y^2)+X^3+Y^3) \cap V(X^3+Y^3-2XY)) = 4$$

since there are no common tangents.

(c) We have

$$Y^5 - X(Y^2-XZ)^2 = 0$$
$$Y^4 + Y^3Z - X^2Z^2 = 0$$

First consider $Y=0$. Then $X^2Z^2=0$ so we have the points $(1:0:0)$ and $(0:0:1)$ (and we can check that both work). Now consider $Y=1$. We have

$$0 = 1 - X(1-XZ)^2 = 1 - X(1-2XZ+X^2Z^2)$$
$$0 = 1 + Z - X^2Z^2$$

Subbing the second into the first we get $1 = X(2-2XZ+Z)$. Solving for Z we get $Z = (1-2X)/(X(1-2X))$ or $1-2X=0$. If $1 \neq 2X$, $X \neq 0$, then

$Z = 1/X$, but this does not satisfy the first equation. If $1 - 2X = 0$ then we get

$$0 = 1 + Z - \frac{Z^2}{4}$$

So $Z = 2 \pm 2\sqrt{2}$ giving the points $(1 : 2 : 4 \pm 4\sqrt{2})$.

Now calculate the intersection multiplicities

$(0 : 0 : 1)$: Dehomogenize with $Z = 1$. Let $P = (0, 0)$. We have

$$\begin{aligned}
&\mathrm{In}(P, V(Y^5 - XY^4 + X^2Y^2 - X^3) \cap V(Y^4 + Y^3 - X^2)) \\
&= \mathrm{In}(P, V(Y^2(Y^3 - 2XY^2 - XY + X^2)) \cap V(Y^4 + Y^3 - X^2)) \\
&= \mathrm{In}(P, V(Y^2) \cap V(-X^2)) + \mathrm{In}(P, V(Y^3 - 2XY^2 - XY + X^2) \cap V(Y^4 + Y^3 - X^2)) \\
&= 4 + \mathrm{In}(P, V(Y^3 - 2XY^2 - XY + X^2) \cap V(Y(Y^3 + 2Y^2 - 2XY - X))) \\
&= 4 + \mathrm{In}(P, V(X^2) \cap V(Y)) + \mathrm{In}(P, V(Y(Y^2 + Y^2X - 2X^2 - X)) \cap V(Y^3 + 2Y^2 - 2XY - X)) \\
&= 6 + 1 + \mathrm{In}(P, V(Y^2 + Y^2X - 2X^2 - X) \cap V(Y^3 - Y^2X + Y^2 + 2X^2 - 2XY)) \\
&= 7 + 2 = 9
\end{aligned}$$

$(1 : 2 : 4 \pm 4\sqrt{2})$: P is a smooth point of both curves so the intersection multiplicity is 1.

$(1 : 0 : 0)$: By Bezout's theorem this point must also have multipliciyty 9.

(d) We have

$$\begin{aligned}
(X^2 + Y^2)^2 + YZ(3X^2 - Y^2) &= 0 \\
(X^2 + Y^2)^3 - 4X^2Y^2Z^2 &= 0
\end{aligned}$$

If $Y = 0$ then $X = 0$ and the point $(0 : 0 : 1)$ works. Likewise if $X = 0$ then $Y = 0$. Now let $X = 1$ and use Maple: get the six points $(1 : \pm i : 0)$, $(1 : \frac{10 \pm \sqrt{80}}{2} : \frac{2}{5}\left(\frac{10 \pm \sqrt{80}}{2}\right)^2 - 2\frac{10 \pm \sqrt{80}}{2})$. Now calculate the intersection multiplicities. First notice

$$\begin{aligned}
&\mathrm{In}(P, V((X^2 + Y^2)^2 + YZ(3X^2 - Y^2)) \cap V((X^2 + Y^2)^3 - 4X^2Y^2Z^2)) \\
&= \mathrm{In}(P, V((X^2 + Y^2)^2 + YZ(3X^2 - Y^2)) \cap V(YZ((3X^2 - Y^2)(X^2 + Y^2) + 4X^2YZ)))
\end{aligned}$$

$(0 : 0 : 1)$: Dehomogenize with $Z = 1$, $P = (0, 0)$. Continuing the above calculation

$$\begin{aligned}
&\mathrm{In}(P, V((X^2 + Y^2)^2 + Y(3X^2 - Y^2)) \cap V((X^2 + Y^2)^3 - 4X^2Y^2)) \\
&= \mathrm{In}(P, V(X^4) \cap V(Y)) + \mathrm{In}(P, V((X^2 + Y^2)^2 + Y(3X^2 - Y^2)) \cap V((3X^2 - Y^2)(X^2 + Y^2) + 4X^2Y)) \\
&= 4 + \mathrm{In}(P, V(Y(4Y(X^2 + Y^2) - 12X^2 + 3Y^2)) \cap V((3X^2 - Y^2)(X^2 + Y^2) + 4X^2Y)) \\
&= 4 + \mathrm{In}(P, V(Y) \cap V(3X^4)) + \mathrm{In}(P, V(4Y(X^2 + Y^2) - 5X^2 + 3Y^2) \cap V((3X^2 - Y^2)(X^2 + Y^2) + 4X^2 \\
&= 4 + 4 + 6 = 14
\end{aligned}$$

$(1 : i : 0)$: Dehomogenize with $X = 1$, $P = (i, 0)$.

$\mathrm{In}(P, V((1+Y^2)^2 + YZ(3-Y^2)) \cap V((1+Y^2)^3 - 4Y^2Z^2))$

$= \mathrm{In}(P, V(1) \cap V(Y)) + \mathrm{In}(P, V((1+Y^2)^2) \cap V(Z))$

$\quad + \mathrm{In}(P, V((1+Y^2)^2 + YZ(3-Y^2)) \cap V((3-Y^2)(1+Y^2) + 4YZ))$

$= 0 + 2 + \mathrm{In}(P, V((1+Y^2)^2 + YZ(3-Y^2)) \cap V((3-Y^2)(1+Y^2) + 4YZ))$

for the last term translate $Y \leftarrow Y - i$ get that the lowest degree terms are in the first case $-8iY + 4iZ$ and in the second case $-4iZ$ so there are no common tangents and P is a smooth point of each. Thus the intersection multiplicity is 3.

$(1 : -i : 0)$: Arguing as above we again get 3.

rest: By Bezout the remaining points each have multiplicity 1.

F, 5-6: Without loss of generality P is the origin. Let f be the lowest degree part of F. If $f_X \neq 0$ then the lowest degree part of F_X is f_X which has degree one less than the degree of f, and hence $m_P(F_X) = m_P(F) - 1$. On the other hand if $f_X = 0$ then the lowest degree part of F_X has degree at least the degree of f and hence $m_P(F_X) > m_P(F) - 1$. Together this gives the result.

F, 5-22: F is irreducible so F and F_X have no common components. So by Fulton's Corollary 1 to Bezout's Theorem we have that

$$\sum_P m_P(F) m_P(F_X) \leq \deg(F)\deg(F_X) = n(n-1)$$

Using the previous result we get

$$\sum_P m_P(F)(m_P(F) - 1) \leq \sum_P m_P(F) m_P(F_X) \leq n(n-1)$$

We get the most multiple points when each are double points. Let m be the number of multiple points of F. Then $2m \leq n(n-1)$ so $m \leq n(n-1)/2$.