Math 821 Combinatorics Notes

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March 21, 2013

1 Schur functions

1.1 Introduction

These are possibly the least natural but most important basis for symmetric functions.

Definition A semistandard Young tableau of shape λ (or column strict tableau) is an assignment of positive integers to the boxes of the Ferrer's diagram of λ such that the entries are strictly increasing down the columns and weakly increasing along the rows.

Example

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 3 & 4 & & \\ 4 & & & \end{bmatrix}$$

The multiplicity of entries are simply a multi-set of integers, so we get a new partition.

Definition If Y is a young tableau write the content of Y, $cont(Y)$, to be the partition $\mu = (c_1, \dots, c_n)$ with c_i the multiplicity of i in Y.

Definition The Schur function indexed by λ is

$$s_\lambda = \sum_{Y \text{semi-standard young tableau of shape} \lambda} x^{cont(Y)}$$

Example We compute $s_{(2,1)}$ by filling the tableau with shape λ.

$$\begin{bmatrix} 1 & 1 \\ 2 & \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 2 & \end{bmatrix}, \dots$$

So $s_{(2,1)} = x_1^2 x_2 + x_1 x_2^2 + 2x_1x_2x_3 + \dots = m_{(2,1)} + 2m_{(1,1,1)}$.

Proposition 1.1. $s_\lambda \in \Lambda(x)$

Proof. Since adjacent transpositions generate all permutations we only need to check that s_λ is symmetric under these. Consider the appearances of $i, i+1$ in a semistandard Young Tableau contributing to s_λ.

$$\begin{bmatrix} & & & & & & & & & & & & & i & i+1 \\ & & & & & & & & & & i & i & i & & \\ & & & & & i & i & i & i & i+1 & i+1 & & & & \\ i & i & i & i+1 & i+1 & i+1 & i+1 & & & & & & & & \\ i+1 & & & & & & & & & & & & & & \end{bmatrix}$$

Now we want to define an involution on the Young tableau of shape λ which swaps the number of is and $(i+1)$s. To define this

- ignore all vertical $\begin{bmatrix} i \\ i+1 \end{bmatrix}$ pairs
- the other entries now look like

$$\underbrace{iiiiii}_{r}\underbrace{(i+1)(i+1)(i+1)(i+1)(i+1)(i+1)}_{s}$$

Then the involution converts it to

$$\underbrace{iiiiii}_{s}\underbrace{(i+1)(i+1)(i+1)(i+1)(i+1)(i+1)}_{r}$$

This remains semistandard because entries above or to the left are in $\begin{bmatrix} i \\ i+1 \end{bmatrix}$ pairs or are unaffected so we are ok. Clearly it is an involution and preserves the shape so we are done.

$\square$

Proposition 1.2. *The* s_λ *are a basis for* $\Lambda(x)$

Proof. Same flavor as last time, observe

$$s_\lambda = \sum_\mu K_{\lambda,\mu} x^\mu$$

where $K_{\lambda,\mu}$ = Kostka number = # of semistandard Young tableau of shape λ and $cont(Y) = \mu$.

The $K_{\lambda,\mu}$ are finite since $cont(Y)$ has at most $|\lambda|$ non-zero entries, so there can only be finitely many in a given partition. Next note that both sides of the observation are symmetric functions. On the right $K_{\lambda,\mu} x^\mu$ appears and on the left x^μ appears as many times as there are semistandard Young tableau of shape λ with content μ, i.e $K_{\lambda,\mu}$ times.

Next, observe that if $\mu \ntrianglelefteq \lambda$ then $K_{\lambda,\mu} = 0$. Suppose we could fill a tableau of shape λ and content μ but $\mu \ntrianglelefteq \lambda$. Then we have for some j that $\lambda_1 - \mu_1 + \ldots + \lambda_j - \mu_j < 0$. Since the columns of a semistandard Young tableau are strict, all copies of $\{1, \ldots, j\}$ will not be in the first j rows. Thus the number of boxes in first j rows is $\lambda_1 + \ldots + \lambda_j \geq \#$ copies of $1, \ldots, j = \mu_1 + \ldots + \mu_j$. This is a contradiction.

Finally, $K_{\lambda,\lambda} = 1$ since

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & & \\ 3 & 3 & 3 & & & & \\ \vdots & & & & & & \end{bmatrix}$$

Is the only possible filling. We now have a triangular system with 1's on the diagonal so as before we are done.

$\square$

1.2 The co-algebra structure on other bases

Definition let μ, λ be partitions say $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for all i.

Observe if $\mu \subseteq \lambda$ then the Ferrer's diagram of μ is contained in the Ferrer's diagram of λ with the top left corners aligned.

Definition If $\mu \subseteq \lambda$ are partitions then the skew Ferrer's diagram λ/μ is the Ferrer's diagram of λ with the boxes of the Ferrer's diagram of μ removed (top left corners aligned).

Example

$$(5,2,1,1)/(3,2,1) = \begin{bmatrix} . & . & . & \square\square \\ . & . & & \\ . & & & \\ \square & & & \end{bmatrix}$$

Ex $(7,5,3,2)/(3,2,2,2)$

$$(7,5,3,2)/(3,2,2,2) = \begin{bmatrix} . & . & . & \square & \square\square\square \\ . & . & \square & \square & \\ . & . & \square & & \\ . & . & & & \end{bmatrix}$$

Note its not necessarily connected. We can have semistandard fillings of these are well with the same constraints.

Definition Given $\mu \subseteq \lambda$ partitions let

$$s_{\lambda/\mu} = \sum_{\text{semistandard fillings } T \text{ of shape } \lambda/\mu} x^{cont(T)}$$

These are called skew-schur functions. These are indeed symmetric (same involution argument).

Proposition 1.3. *:*

(a) $\Delta p_n = 1 \otimes p_n + p_n \otimes 1$. *Indeed they are all primitive.*

(b) $\Delta e_n = \sum_{i+j=n, i,j \geq 0} e_i \otimes e_j$

(c) $\Delta h_n = \sum_{i+j=n, i,j \geq 0} h_i \otimes h_j$

(d) $\Delta s_\lambda = \sum_{\mu \subseteq \lambda} s_\mu \otimes s_\lambda$

Proof. (a) $\Delta p_n = \sum x_i^n + \sum y_i^n = p_n(x) + p_n(y) = p_n \otimes 1 + 1 \otimes p_n$

(b)

$$\Delta e_n = \sum_{i_1 < \ldots < i_n} (x's \text{ and } y's) = \sum_{i=0}^{n} \left(\sum_{i_1 < \ldots < i_n} (x_{i_1} \ldots x_{i_k}) \right) \left(\sum_{i_1 < \ldots < i_n} (y_{i_1} \ldots y_{i_k}) \right)$$

$$\sum_{k=0}^{n} e_k(x) e_{n-k}(y)$$

(c) same as above

(d) Observe that since we're working with symmetric functions we can take any total order on the positive integers to give the semistandard conditions in the tableaux. It doesn't need to be the same order type. So

$$\Delta s_\lambda = s_\lambda(x, y)$$

Choose the ordering $x_1 < x_2 < \ldots < y_1 < y_2 < \ldots$. Then in filling in a shape λ the xs will always appear above and to the left of any ys in every row and column. So the partition of the diagram filled with the xs is itself the Ferrer's diagram of a partition μ and the ys are in λ/μ. In both cases satisfying the semi-standard condition.

Conversely, given $\mu \subseteq \lambda$ those fillings of λ with x's in μ and y's in λ/μ are valid fillings. So $\Delta s_\lambda = \sum_{\mu \subseteq \lambda} s_\mu \otimes s_\lambda$

□

Now we're ready to sort out h_n.

Proposition 1.4. *The* $\{e_n\}_n$ *and* $\{h_n\}_n$ *are algebraically independent and generate* $\Lambda(x)$ *as a polynomial algebra. Furthermore, if char* $k = 0$ *this is true of* $\{p_n\}_n$ *as well.*

Proof. We know the e_λ and p_λ are bases but they themselves are monomials in the e_n, p_n. so this gives the result in this case. For h_n consider the generating functions

$$H(t) = \sum_{n \geq 0} h_n(x) t^n = \text{"}MSet(x)\text{"} = \prod_{i=1}^{\infty} \frac{1}{1 - x_i t}$$

$$E(t) = \sum_{n \geq 0} e_n(x) t^n = "PSet"(x) = \prod_{i=1}^{\infty} (1 + x_i t)$$

So

$$H(t)E(-t) = \prod_{i=1}^{\infty} \frac{1 - x_i t}{1 - x_i t} = 1$$

and by co-efficient extraction

$$[t^n]H(t)E(-t) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases} = \sum_{i+j=n, i,j \geq 0} (-1)^j h_i(x) e_j(x)$$

So we have a system of equations which we can solve for either e_n or h_n.

$$\begin{aligned} e_0 &= h_0 = 1 \\ &\left.\begin{aligned} e_n &= e_{n-1}h_1 - e_{n-2}h_2 + \dots \\ h_n &= h_{n-1}e_2 - h_{n-2}e_2 + \dots \end{aligned}\right\} (*) \end{aligned}$$

these have the same shape . We know the $\{e_n\}$ *are algebraically independent generators, so define*

$$\begin{aligned} \omega : \Lambda(x) &\to \Lambda(x) \\ e_n &\to h_n \end{aligned}$$

since the e_n *are independent generators this is a well defined endomorphism. Since the recurrences* $(*)$ *are the same we have* $\omega \circ \omega = id$*. So* ω *is an involution and the* $\{h_n\}$ *are independent algebra generators.*

Remark ω *is called the* fundamental involution

□

2 References

Reiner 2.2, 2.3, 2.4