Notes on Local Cohomology

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1 Homological Algebra

The aim of this section is to give basic definitions of category theory, some constructions like derived functors, spectral sequences and prove some important results. All of this can be found in Chapter 1 of [4].

1.1 Categories and Functors

This section is intended to give (and hence fix) some notations as well as definitions. We will define various familiar looking terms (like injective, surjective morphisms, subobject, kernel, cokernel etc.) in context of any general category. However, to make sense of direct limits, exact sequences etc. we will need to put additional axioms and hence define additive and abelian categories.

Definition 1 A category $\mathscr{C}$ consists of a class $Obj(\mathscr{C})$ (objects of category) and for every two objects $A, B \in Obj(\mathscr{C})$, a set $Hom_{\mathscr{C}}(A, B)$ (morphisms in category) together with map (called composition), for every $A, B, C \in \mathscr{C}$

$$Hom_{\mathscr{C}}(A, B) \times Hom_{\mathscr{C}}(B, C) \longrightarrow Hom_{\mathscr{C}}(A, C)$$

satisfying following axioms:

(C1) For $A, B, A', B' \in Obj(\mathscr{C})$, with $(A, B) \neq (A', B')$, $Hom_{\mathscr{C}}(A, B)$ and $Hom_{\mathscr{C}}(A', B')$ are disjoint sets.

(C2) There exists $1_A \in Hom_{\mathscr{C}}(A, A)$, which acts as identity with respect to composition map (i.e, for every $u \in Hom_{\mathscr{C}}(A, B)$ and $v \in Hom_{\mathscr{C}}(B, A)$, we have $u1_a = u$ and $1_A v = v$), called identity morphism.

(C3) Composition is an associative map. That is, for every $u \in Hom_{\mathscr{C}}(A, B)$, $v \in Hom_{\mathscr{C}}(B, C)$ and $w \in Hom_{\mathscr{C}}(C, D)$, we have $(wv)u = w(vu)$.

For simplifying notations, we will use just $\mathscr{C}$, in place of $Obj(\mathscr{C})$ and $Hom(A, B)$ in place of $Hom_{\mathscr{C}}(A, B)$, when there is no risk of confusion.

Definition 2 Let $\mathscr{C}$ and $\mathscr{C}'$ denote two categories. A map $F : \mathscr{C} \longrightarrow \mathscr{C}'$ is called *covariant functor* if it associates to each object $A \in \mathscr{C}$ an object $F(A) \in \mathscr{C}'$ and to each morphism $u \in Hom_{\mathscr{C}}(A, B)$ a morphism $F(u) \in Hom_{\mathscr{C}'}(F(A), F(B))$ such that following axioms hold:

(F1) For any $A \in \mathscr{C}$, $F(1_A) = 1_{F(A)}$.

(F2) For any $A, B, C \in \mathscr{C}$ and $u \in Hom_{\mathscr{C}}(A, B)$, $v \in Hom_{\mathscr{C}}(B, C)$ we have $F(vu) = F(v)F(u)$

In similar fashion, we can define *contravariant functor*, where arrows are reversed. That is, if $u \in Hom_{\mathscr{C}}(A, B)$, we have $F(u) \in Hom_{\mathscr{C}'}(F(b), F(A))$ and in this case $F2$ is modified accordingly.

Definition 3 Let F, F' be two functors from $\mathscr{C}$ to $\mathscr{C}'$. A *natural transformation* between two such functors is a map $\eta : F \to F'$, which assigns to each $A \in \mathscr{C}$ a morphism $\eta(A) : F(A) \to F'(A)$, such that for each $u \in Hom_{\mathscr{C}}(A, B)$ the following diagram is commutative:

$$\begin{tikzcd} F(A) \arrow[r, "\eta(A)"] \arrow[d, "F(u)"'] & F'(A) \arrow[d, "F'(u)"] \\ F(B) \arrow[r, "\eta(B)"'] & F'(B) \end{tikzcd}$$

We can make sense of injective, surjective, bijective morhpisms as well as isomorphisms in context of general categories. Consider a category $\mathscr{C}$ and $A, B \in \mathscr{C}$. Let $u \in Hom(A, B)$. This morphism defines two maps for any other object $C \in \mathscr{C}$

$$\begin{array}{ccc} Hom(B, C) & \longrightarrow & Hom(A, C) \\ v & \longmapsto & vu \end{array} \tag{1}$$

$$\begin{array}{ccc} Hom(C, A) & \longrightarrow & Hom(C, B) \\ w & \longmapsto & uw \end{array} \tag{2}$$

We define u to be injective (resp. surjective) if for every $C \in \mathscr{C}$, the corresponding (2) (resp. (1)) is an injective map. A morphism which is both injective and surjective is called bijective morphism. u is called isomorphism if it has both left and right inverses. This means there exist $v, w \in Hom(B, A)$ such that $vu = I_A$ and $uw = 1_B$. It should be noted that an isomorphism is always a bijective morphism, whereas the converse need not be true.

Now for a fixed object $A \in \mathscr{C}$, consider the class of injective maps $\{u \in Hom(A', A) : A' \in \mathscr{C}\}$. We can put ordering on this class by saying $(A', u') \leq (A'', u'')$ if there is a morhpism $v \in Hom(A', A'')$ such that $vu'' = u'$ (this is termed as (A', u') is extended by $A'', u'')$. We say two elements of this class are equivalent if each one is extended by the other. An equivalence class of this class is called subobject of A. It should be noted that a subobject of A is not just an object of $\mathscr{C}$. In principle, it is a pair (A', u'), representing an equivalence class defined. It is easy to check that two objects in same equivalence class are naturally isomorphic. Therefore, we can justifiably abuse the notation to say A' is subobject of A, with injective morphism $A' \to A$ being understood.

The dual notion to subobject is *quotient object*. One can attempt to define them by just reversing arrows and replacing injective morphisms by surjective morphisms. We will omit the explicit definition, since it is obvious.

Now assume we are given a family of morphisms $\{u_i \in Hom(A_i, A)\}_i$ (resp. $\{v_i \in Hom(A, A_i)\}_i$). We clearly have the following two set maps.

$$\begin{array}{rcl} Hom(A,B) & \longrightarrow & \prod_i Hom(A_i, B) \\ w & \longmapsto & (wu_i)_i \end{array} \tag{3}$$

$$\begin{array}{rcl} Hom(B,A) & \longrightarrow & \prod_i Hom(B, A_i) \\ w & \longmapsto & (v_i w)_i \end{array} \tag{4}$$

We say A is represented as *direct sum* of family $\{u_i\}_i$ (resp. *direct product* of family $\{v_i\}_i$) if for any object $B \in \mathscr{C}$, the corresponding maps (3) (resp. (4)) is a bijective map. Then we write $A = \oplus_i A_i$ (resp. $A = \prod_i A_i$), with morphisms u_i (resp. v_i) being understood and called *canonical injections* (resp. *canonical projections*). It is easily verified that direct sum and direct product of a finite family of objects are isomorphic (if they exist).

Definition 4 A category $\mathscr{C}$ is called additive if

(A1) For every $A, B \in \mathscr{C}$, $Hom(A, B)$ has structure of abelian group and composition is bilinear map.

(A2) Finite direct sums exist in $\mathscr{C}$.

(A3) There is a *zero object* characterized by property that $1_{zero} = 0$. We denote this object by $0_{\mathscr{C}}$, since any other object with same property is isomorphic to it.

We can define kernel and cokernel of a morphism in case of additive category. Let $u \in Hom(A, B)$ be a morphism in $\mathscr{C}$ (assumed to be additive). *Generalized kernel* of this morphism is subobject (A', i) of A, such that $ui = 0$ and any morphism $v \in Hom(C, A)$, with $uv = 0$ factors through A'. This is denoted by $Ker(u)$. One can similarly define *cokernel*, denoted by $Coker(u)$. Now image of this morphism ($Im(u)$) is kernel of cokernel and coimage ($Coim(u)$) is cokernel of kernel. Note that none of these necessarily exist, but if they do, we get a natural map, $\overline{u} : Coim(u) \to Im(u)$, such that u is same as composition $A \to Coim(u) \to Im(u) \to B$, where end morphisms are canonical.

In case of additive categories, we define (and usually work with) *additive functors*, defined as functor $F : \mathscr{C} \to \mathscr{C}'$ between two additive categories, such that for any two $u, v \in Hom_{\mathscr{C}}(A, B)$, we have $F(u + v) = F(u) + F(v)$.

Definition 5 An additive category $\mathscr{C}$ is called *abelian category*, if it satisfies the additional two axioms:

(AB1) Kernel and cokernel exist for every morphism in $\mathscr{C}$.

(AB2) If u is a morphism in $\mathscr{C}$, then the canonical map $\overline{u} : Coim(u) \to Im(u)$ is an isomorphism.

From now on, fix $\mathscr{C}$ to be abelian category.

We can make sense of exact sequence of morphisms in abelian category. Just for sake of completeness, a sequence of morphisms $A \xrightarrow{u} B \xrightarrow{v} C$ is called exact if $Ker(v) = Im(u)$.The usual definitions of *exact* (resp. *left exact*, *right exact*) (covariant) functors, can also be framed in case of abelian categories. Assume $0 \longrightarrow A \xrightarrow{u} B \xrightarrow{v} C \longrightarrow 0$ be an exact sequence of morphisms in $\mathscr{C}$. Let $F : \mathscr{C} \longrightarrow \mathscr{C}'$ be (additive) functor between two abelian categories. We say F is *exact* (resp. *semi-exact*, *left exact* and *right exact*) if $0 \longrightarrow F(A) \xrightarrow{F(u)} F(B) \xrightarrow{F(v)} F(C) \longrightarrow 0$ (resp. $F(A) \xrightarrow{F(u)} F(B) \xrightarrow{F(v)} F(C)$, $0 \longrightarrow F(A) \xrightarrow{F(u)} F(B) \xrightarrow{F(v)} F(C)$ and $F(A) \xrightarrow{F(u)} F(B) \xrightarrow{F(v)} F(C) \longrightarrow 0$) is exact.

We need to put some additional axioms which are not assumed in definition of abelian category. These axioms will be mentioned explicitly, if assumed to hold in an abelian category, in rest of lectures.

(AB3) Direct sum of arbitrary family of morphisms exist in $\mathscr{C}$

(AB4) Direct sum of injective morphisms is an injective morphism.

(AB5) If $A \in \mathscr{C}$ and $(A_i)_i$ is family of subobjects of A, then for any subobject B of A,

$$inf(sup_i(A_i), B) = sup_i(inf(A_i, B))$$

where, we have following definition:
If (A_i) is family of subobjects of A, then $sup(A_i) := Im(\oplus_i A_i \to A)$. If P, Q are subobjects of A, then $inf(P, Q) := Ker(A \to A/P \times A/Q)$

1.2 Category of Diagrams

Definition 6 A sheme of diagrams is a graph $\sum = (V, E, d)$, where V is set of vertices, E is set of edges and $d : E \longrightarrow V \times V$ is assigning direction to each edge.

A *diagram* D in an additive category $\mathscr{C}$ is a map such that $D(v) \in \mathscr{C}$ to every vertex $v \in V$ and for every $\varphi \in E$ such that $d(\varphi) = (v, w)$, $D(\varphi) \in Hom_{\mathscr{C}}(D(v), D(w))$. We can also specify certain commutation relations to hold (if necessary).

Definition 7 Given a scheme of diagrams $\sum$ and $v, w \in V$, we define *path* starting at v and ending at w to be sequence of edges $\tilde{\varphi} = (\varphi_0, \ldots, \varphi_n)$, such that end of $d(\varphi_i)$ is same as begining of $d(\varphi_{i+1})$ and starting vertex of $d(\varphi_0) = v$ and end vertex of $d(\varphi_n) = w$. Let $P(v, w)$ denote the set of paths from v to w. Let $R(v, w)$ denote a subset of formal combination of paths (from v to w) with integer coefficients. That is, for any $r \in R(v, w)$,

$$r = \sum_{finite} n_i \tilde{\varphi}_i \; where \; n_i \in \mathbb{Z}, \; \tilde{\varphi}_i \in P(v, w) \; \forall i$$

For $v \in V$, $R(v,v)$ will have an additional element denoted by e_v. Then $R = \cup_{v,w\in V}$ is called *relations of commutation*.

Now, a diagram D in additive category $\mathscr{C}$ is said to satisfy relation of commutation R is for every $r = \sum n_i \tilde{\varphi}_i \in R$, we have $\sum n_i D(\tilde{\varphi}_i) = 0$. Here we define $D((\varphi_0, \dots, \varphi_n)) := D(\varphi_0) \dots D(\varphi_n)$ by composition and $D(e_v)$ is replaced by $Id_{D(e_v)}$.

Assume there are two diagrams D, D' in $\mathscr{C}$ satisfying relations of commutation R. A morphism between them is a map η such that for every $v \in V$, $\eta(v) \in Hom(D(v), D'(v))$, such that following diagrams commute for every $\varphi \in E$, $d(\varphi) = (v, w)$ where $v, w \in V$.

$$\begin{tikzcd} D(v) \arrow[r, "\eta(v)"] \arrow[d, "D(\varphi)"'] & D'(v) \arrow[d, "D'(\varphi)"] \\ D(w) \arrow[r, "\eta(w)"'] & D'(w) \end{tikzcd}$$

Thus we can talk of *category of diagrams* associated to every scheme with commutation relations $(\sum, R)$, with values in $\mathscr{C}$, denoted by $\mathscr{C}^{(\sum,R)}$, whose objects are diagrams satisfying relations of commutation R and morphisms are morphisms of diagrams. The next result shows that usual properties of $\mathscr{C}$ are carried over $\mathscr{C}^{(\sum,R)}$.

Theorem 1 *Let $(\sum, R)$ be a scheme of diagrams with commutation relations R and let $\mathscr{C}$ be additive category. Then $\mathscr{C}^{(\sum,R)}$ is an additive category. If $\mathscr{C}$ is abelian category (resp. satisfies axioms* (AB3), (AB4), (AB5)*), then so is true for $\mathscr{C}^{(\sum,R)}$.*

The proof, omitted here, consists of taking kernel, cokernel etc. at each vertex.

Example 1 One important example of category of diagrams is given by taking a preordered set I as set if vertices and $\{(i,j) : i \leq j\}$ as set of edges. Define relations of commutation to be $(i,j)(j,k) = (i,k)$ and $(i,i) = e_i$. We denote this scheme of diagrams by I again. Then $\mathscr{C}^I$ has *inductive systems* of objects of $\mathscr{C}$ as objects. For a given topological space X, if we replace I by set of open subspaces of X and take inverse order relation to be preorder, $\mathscr{C}$ is *category of presheaves* over X with values in $\mathscr{C}$.

1.3 Inductive Limits

Let I be a preordered set and denote the scheme of diagrams defined by I (as in example of previous section) by I again. For any category $\mathscr{C}$, $\mathfrak{A} \in \mathscr{C}^I$ is called *inductive system with values in* $\mathscr{C}$. Let $\mathfrak{A} = (A_i, u_{ij})$, where A_i is object of $\mathscr{C}$ at vertex i and $u_{ij} \in Hom_{\mathscr{C}}(A_i, A_j)$. Such an inductive system is said to have *inductive (or direct) limit*, denoted by $\varinjlim \mathfrak{A}$ or $\varinjlim A_i$, is an object $A \in \mathscr{C}$ together with morphisms $u_i \in Hom_{\mathscr{C}}(A_i, A)$ such that following conditions are satisfied:

1. For every $i \leq j$ the following diagram commutes

$$\begin{tikzcd} A_i \arrow[rr, "u_{ij}"] \arrow[dr, "u_i"'] & & A_j \arrow[dl, "u_j"] \\ & A & \end{tikzcd}$$

2. If there exists any object $B \in \mathscr{C}$ with morphisms $v_i \in Hom_{\mathscr{C}}(A_i, B)$ such that for every $i \leq j$ the following diagram commutes

$$\begin{tikzcd} A_i \arrow[rr, "u_{ij}"] \arrow[dr, "u_i"'] & & A_j \arrow[dl, "u_j"] \\ & B & \end{tikzcd}$$

   Then there is a unique map $\varphi \in Hom_{\mathscr{C}}(A, B)$, such that for every i the following diagram commutes

$$\begin{tikzcd} & A_i \arrow[dl, "u_i"'] \arrow[dr, "v_i"] & \\ A \arrow[rr, "\varphi"'] & & B \end{tikzcd}$$

Of course such an object need not exist, but if for every inductive system, we can form inductive limits, then this is a well defined (covariant) functor $\varinjlim : \mathscr{C}^I \longrightarrow \mathscr{C}$. The next theorem states condition under which we can hope for existence of direct limits and exactness properties of this functor (assuming of course that underlying category is abelian).

Theorem 2 *If $\mathscr{C}$ is abelian category satisfying axiom (AB3) and I is any preordered set, then for any $\mathfrak{A} \in \mathscr{C}^I$, $\varinjlim \mathfrak{A}$ exists and in this case $\varinjlim$ is right exact, additive functor. If in addition $\mathscr{C}$ satisfies axiom (AB5), this functor is exact.*

PROOF:
We give construction of direct limit and leave the verifications. Assume that directed system $\mathfrak{A} = (A_i, u_{ij})$ be given. Let $S = \bigoplus_i A_i$. Consider the mappings $A_i \to A_i + A_j$ [1], induced by id and $-u_{ij}$ for each $i \leq j$. Let $N_{ij} = Im(A_i \to S)$. Again take $N = sup_{ij}(N_{ij})$. Then $\bar{A} = S/N$ is direct limit of the given system.

QED

[1] sup of objects is also written as sum

1.4 Generators in a Category

Definition 8 A family of objects $(U_i)_{i\in I}$ of some category $\mathscr{C}$ is said to be *family of generators* if for every $A \in \mathscr{C}$ and subobject $A' \in \mathscr{C}$ of A, $A' \neq A$ we can find $i \in I$ and $u \in Hom(U_i, A)$ such that there doesn't exist any $v \in Hom(U_i, A')$ with $jv = u$ (here j denotes canonical inclusion of A' in A).

Now assume that $\mathscr{C}$ is abelian category satisfying axiom (AB3). Then if $(U_i)_{i\in I}$ is family of generators of $\mathscr{C}$, then obviously $U = \oplus_i U_i$ is a generator. In this case, for any $A \in \mathscr{C}$, we can take $S = Hom(U, A)$ and construct direct sum $\oplus_{f\in S} U_f$, where each U_f is same object as U. Consider the following morphism induced by $f : U_f \longrightarrow A$

$$\bigoplus_{f\in S} U_f \longrightarrow A$$

This map is clearly surjective and hence identifies A as quotient object of some direct sum of copies of generator. We record these observations in following lemma:

Lemma 1 *Let $\mathscr{C}$ be abelian category satisfying axiom* (AB3)*. Then the following conditions are equivalent:*

1. *$(U_i)_{i\in I}$ is a family of generators of $\mathscr{C}$.*
2. *$U = \oplus_i U_i$ is a generator of $\mathscr{C}$.*
3. *Every object of $\mathscr{C}$ is quotient of some direct sum of copies of U.*

Another useful observation is to construct generator for $\mathscr{C}^{\sum}$ for some scheme of diagrams $\sum = (V, E, d)$ (with or without commutation relations), from family of generators of $\mathscr{C}$. For every $i \in I$ and $s \in V$, let $\mathscr{E}_{i,v}$ denote the diagram such that

$$\mathscr{E}_{i,v}(w) = \begin{cases} U_i & if \quad v = w \\ 0 & if \quad v \neq w \end{cases}$$

and all morphisms $\mathscr{E}_{i,v}(\varphi)$ are zero. Then it can be verified trivially that $\{(\mathscr{E}_{i,v})_{i\in I,\, v\in V}\}$ forms family of generators for corresponding category of diagrams.

1.5 Injective Resolutions

This section is the heart of all what follows. All the constructions in next (or probably every) section from now onwards will be depending on the fact that in the relevant category, injection resolutions do exist. We fix $\mathscr{C}$, an abelian category.

Definition 9 $M \in \mathscr{C}$ is called *injective*, if corresponding (covariant) $Hom(-, M)$ functor is exact. In other words, if $\mathscr{A}$ denote the category of abelian groups, then the following functor is exact

$$\begin{array}{ccc} \mathscr{C} & \longrightarrow & \mathscr{A} \\ A & \longmapsto & Hom(A, M) \end{array}$$

This is just another way of saying that if B is subobject of some object A, then every map from B to M can be extended to map from A to M.

Definition 10 A morphism $A \to M$ (beware, M is not assumed to be injective object here) is called *injective envelop* of A, if it is injective morphism and for every injective morphism $B \to C$ and some morphism $B \to A$, one can find another $C \to M$, making following diagram commutative.

$$\begin{tikzcd}
B \arrow[r] \arrow[d] & C \arrow[d] \\
A \arrow[r] & M
\end{tikzcd}$$

To prove main result, Theorem 3, we need a lemma stating to check whether some object is injective, it is enough to check for generator (if it exists)

Lemma 2 *Assume $\mathscr{C}$ admits a generator, say U. Then $M \in \mathscr{C}$ is injective object if and only if for every subobject V of U and a morphism $V \to M$, there is a morphism $U \to M$, which extends it.*

Proof:
It suffices to prove sufficiency of this condition. Let B be subobject of A and we are given morphism $u : B \to M$, where M is an object satisfying condtion of lemma. We need to find extension of u $A \to M$ to prove that M is injective. Let $\mathscr{P}$ be set of subobjects of A for which such extension exists. Then by Zorn's lemma (this set is inductive, by axiom (AB5)), we can find maximal element (say $B', u')$) of $\mathscr{P}$. Assume that $B' \neq A$. Then there exists $j : U \to A$ such that $Im(j) \not\subset B'$. Let $V = j^{-1}(B')$ and consider the following:

$$\begin{tikzcd}
V \arrow[r, "u'j"] \arrow[d, hook] & M \\
U \arrow[ur, "\tilde{u}"'] &
\end{tikzcd}$$

where $\tilde{u}$ is extension, which exists by assumption on M. Now let $B'' = B' + j(U)$. We have the following diagram (the row is exact and dotted map is to be constructed):

$$\begin{tikzcd}
V \arrow[r, "(\iota,-j)"] & U \oplus B' \arrow[r, "(j,\iota)"] \arrow[d, "(\tilde{u},u')"] & B'' \arrow[r] & 0 \\
 & M & &
\end{tikzcd}$$

Thus B'' can be identified with quotient of $U \oplus B'$ and to define map from B'' to M, we only need to check that $Ker(U \oplus B' \to B'') \subseteq Ker(U \oplus B' \to M)$, where the first term is same as V. This is obvious since $V \to U \oplus B' \to M$ composition is zero map. Thus we have extended u to $B'' \neq B'$ and containing B' which contradicts maximality of B'.

QED

Theorem 3 *If $\mathscr{C}$ satisfies (AB5) and admits a generator, then for every $A \in \mathscr{C}$, there is an injective morphism of A into some injective object M.*[2]

1.6 Derived Functors

We are only concerned here with covariant functors.

Definition 11 Let $\mathscr{C}$ be an abelian category and $\mathscr{C}'$ be some additive catogory. For any two numbers a and b (which can be ∞ or $-\infty$), such that $a+1 < b$, a *covariant ∂ functor* from $\mathscr{C}$ to $\mathscr{C}'$ of degrees $a < i < b$ is a system $T = (T^i)_{a<i<b}$ of additive covariant functors from $\mathscr{C}$ to $\mathscr{C}'$ and for every i, $a < i < b-1$ and exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathscr{C}$, a morphism

$$\partial : T^i(A'') \longrightarrow T^{i+1}(A')$$

satisfying following axioms:

1. If we have following commutative diagram with both rows exact

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0 \end{tikzcd}$$

   Then the following diagram is commutative:

$$\begin{tikzcd} T^i(A'') \arrow[r, "\partial"] \arrow[d] & T^{i+1}(A') \arrow[d] \\ T^i(B'') \arrow[r, "\partial"] & T^{i+1}(B') \end{tikzcd}$$

2. For every exact sequence $0 \to A' \to A \to A'' \to 0$, the corresponding long sequence

$$\ldots T^i(A') \to T^i(A) \to T^i(A'') \to T^{i+1}(A') \to \ldots \tag{5}$$

   is a complex (that is, composition of every two consecutive morphisms is zero.

[2]this is equivalent to saying injective resolution exists, which by definition, is exact sequence $0 \to A \to I_0 \to I_1 \to \ldots$ where each I_i is injective object.

If $\mathscr{C}'$ is abelian category and (5) is exact sequence, then corresponding $\partial - functor$ is called *cohomology functor*. Assume $T = (T^i)$ and $T' = (T'^i)$ are two ∂ functors of same degrees, then we have notion of morphism between these two ∂-functors, which is a system of natural transformations $f = (f^i)$, where each f^i is natural transformation between T^i and T'^i, such that for each $0 \to A' \to A \to A'' \to 0$, exact sequence in $\mathscr{C}$, the following diagram commutes:

$$\begin{tikzcd}
T^i(A'') \arrow[r, "\partial"] \arrow[d] & T^{i+1}(A') \arrow[d] \\
T'^i(A'') \arrow[r, "\partial"'] & T'^{i+1}(A')
\end{tikzcd}$$

Now a *universal ∂ functor* is a ∂ functor $T = (T^i)_{0 \le i \le a}$, such that for every $T' = (T'^i)$ of same degrees, and a natural transformation $f : T^0 \to T'^0$, there is a unique morphism of ∂ functors $\tilde{f}$ between T and T' which is same as f in degree 0. By this definition, given any functor $F : \mathscr{C} \to \mathscr{C}'$, there can be atmost one (upto isomorphism), universal ∂ functor with degrees $0 \le i \le a$, which is same as F in degree 0. Its higher degrees are denoted by S^iF and called *satellite functors of F*. Of course, it is not obvious that such sequence of functors exist. It will be proved in Theorem 4.

To prove the main result, Theorem 4, we need notion of *effacability*. First observe that to prove the existence of satellite functors in all degrees, it is enough to prove existence of S^1F and $S^{-1}F$, by virtue of the formulae $S^1(S^iF) = S^{i+1}F$ and $S^{-1}(S^iF) = S^{i-1}F$.

Definition 12 An additive functor from $\mathscr{C}$ (assumed abelian) to $\mathscr{C}'$ (assumed additive) is called *effacable* if for every $A \in \mathscr{C}$, one can find an injective morphism $u : A \to M$ such that $F(u) = 0$.

Now we can state characterisation of univeral ∂ functor in terms of effacability.

Proposition 1 *Let $\mathscr{C}$ and $\mathscr{C}'$ be two abelian categories and $T = (T^i)$, $0 < i < a$ be covariant, exact, ∂ functor in degrees $0 < i < a$, where $a > 0$. If T^i is effacable for every $i > 0$, then T is universal and converse holds if every object of $\mathscr{C}$ admits injective envelop.*

PROOF:
Let $T' = (T'^0, T'^1)$ be another ∂ functor, defined in degrees 0 and 1. Assume that we are given f^0, a natural transformation of T^0 into T'^0. We will prove that there is only on way to define $f^1 : T^1 \to T'^1$ (and hence T is universal), assuming that T^1 is effacable.

Let $0 \to A \to M \to A' \to 0$ be exact sequence in $\mathscr{C}$ where $T^1(A) \to T^1(M)$ is zero morphism. Consider the following diagram where first row is exact and second is a complex.

$$\begin{tikzcd}
T^0(M) \arrow[r] \arrow[d, "f^0(M)"'] & T^0(A') \arrow[r] \arrow[d, "f^0(A')"] & T^1(A) \arrow[r] \arrow[d, dashed] & 0 \\
T'^0(M) \arrow[r] & T'^0(A') \arrow[r] & T'^1(A) &
\end{tikzcd}$$

Since $T^0(A') \to T^1(A)$ is surjective, $T^1(A) = T^0(A')/Z$, where $Z = Ker(T^0(A') \to T^1(A))$. We have the composite map $T^0(A') \to T'^0(A') \to T'^1(A)$, denote this by f. In order to define map $T^1(A) \to T'^1(A)$, such that last square of above diagram becomes commutative, the required map must be the following factorization (proving uniqueness):

$$\begin{tikzcd}
T^0(A') \arrow[rr, "f"] \arrow[dr] & & T'^1(A) \\
& T^1(A) = T^0(A')/Z \arrow[ur] &
\end{tikzcd}$$

To prove that such a map can be defined, we only need to check that $f(Z) = 0$, which is true since, $Z = Im(T^0(M) \to T^0(A'))$ and because last row of above commuting diagram is complex. This proves first part of proposition. The second part is essentially contained in next theorem.

QED

Theorem 4 *Let $\mathscr{C}$ be an abelian category in which every object admits injective envelop* [3]*, then for every additive covariant functor on $\mathscr{C}$, the satellite functors S^iF $(i \geq 0)$ exist and are effacable for each $i > 0$. Moreover universal ∂ functor $(S^iF)_{i\geq 0}$ is exact if and only if F satisfies following condition: F is semi-exact and for every $P \subset Q \subset R$ in $\mathscr{C}$,*

$$Ker(F(Q/P) \to F(R/P)) \subseteq Im(F(Q) \to F(Q/P))$$

(it is easy to check that these conditions hold if F is left exact or right exact).

Proof:
As always, we give construction of S^1F and leave details. Let $A \in \mathscr{C}$ and consider the exact sequence: $0 \to A \to M \to A' \to 0$, where $F(A) \to F(M)$ is zero morphism. Now define $S^1F(A) := F(A')/Im(F(M) \to F(A'))$.

QED

Now we are in position to define *derived functors*. Let $\mathscr{C}$ and $\mathscr{C}'$ be two abelian categories, where $\mathscr{C}$ has enough injectives. For any $A \in \mathscr{C}$, consider its injective resolution

[3] this holds if category has enough injectives

$0 \to A \to I^0 \to I^1 \to \ldots$. Then *derived functors of* F [4] are defined as $R^iF(A) = H^i(F(\mathbb{I}))$, where $\mathbb{I}$ stands for the complex $0 \to I^0 \to I^1 \to \ldots$.

Some observations worth mentioning here are:

- By assuming $R^iF = 0$ for $i < 0$, we get ∂ functor $RF = (R^iF)_{-\infty<i<\infty}$. This functor is cohomology functor.
- By universal property of satellite functors, there is unique morphism of ∂ functors $(S^iF) \to R^iF$ [5]. This morphism is isomorphism if and only if F is left exact.

Therefore, if we assume that F is left exact, covariant functor, it make sense to identify its derived functors with universal ∂ functors. We say an object $A \in \mathscr{C}$ is F-*acyclic*, if for each $i > 0$, $R^iF(A) = 0$. The next lemma is useful tool to compute derived functors. It states that any class of objects of $\mathscr{C}$, satisfying some conditions, can be used to form resolution, which will give same derived functors, as is done by injective resolutions. Observe that if $0 \to A \to J^0 \to J^1 \to \ldots$ is an exact sequence, where each C^i are F-acyclic, then $H^i(F(\mathbb{J})) = R^iF(A)$. The following lemma essentially gives some properties, which imply that a certain class of objects in $\mathscr{C}$ has only F-acyclic objects.

Lemma 3 *Let F be covariant functor from an abelian category $\mathscr{C}$ to another abelian category $\mathscr{C}'$, and suppose $\mathscr{C}$ has enough injectives. Let $\mathscr{M}$ be a class of objects of $\mathscr{C}$ satisfying following conditions:*

1. *For every $A \in \mathscr{C}$, there exists an injective morphism from A to M, $M \in \mathscr{M}$.*
2. *If $A \oplus B$ is in $\mathscr{M}$ then both A and B are in $\mathscr{M}$.*
3. *For every exact sequence $0 \to M' \to M \to M'' \to 0$, with $M', M \in \mathscr{M}$, we have $M'' \in \mathscr{M}$ and in this case $0 \to F(M') \to F(M) \to F(M'') \to 0$ is exact sequence.*

Then every injective object is in $\mathscr{M}$ and every $M \in \mathscr{M}$ is F-acyclic.

Proof:
Let I be some injective object in $\mathscr{C}$. Then by first assumption, there is some $M \in \mathscr{M}$ and an injective morphism $v : I \to M$. Since I is injective, v admits a section (that is, there is a morphism $u : M \to I$ such that $uv = 1_I$) which identifies I as direct factor of M. Then by second assumption I is in $\mathscr{M}$.

Let $M \in \mathscr{M}$ and $0 \to M \to I^0 \to \ldots$ be injective resolution of M. Let $Z^i = Ker(I^i \to I^{i+1})$. To prove that $R^pF(M) = 0$ for each p, it suffices to prove that for each $i \geq 0$, the sequence $0 \to F(Z^i) \to F(I^i) \to F(Z^{i+1}) \to 0$ is exact and $0 \to F(M) \to F(I^0) \to$

[4] We are here concerned with covariant functors only and what we call derived functor is *right* derived functor, as defined in literature. Since it is only matter of reversing arrows, we wont define left derived functors of contravariant functor here.

[5] since in degree 0, $S^0F(A) = F(A)$, we have $F(A) \to Ker(F(I^0) \to F(I^1)) = R^0F(A)$

$F(Z^1) \to 0$ is exact. The later sequence is exact since $M, I^0 \in \mathscr{M}$ and we invoke third assumption. The first claim is proved by induction on i. By virtue of exact sequences $0 \to Z^i \to I^i \to Z^{i+1} \to 0$, we inductively prove that each Z^i is in $\mathscr{M}$. This will suffice because of third assumption and the fact that each $I^i \in \mathscr{M}$. Now $Z^0 \cong M$, and Z^1 is third term of exact sequence $0 \to M \to I^0 \to Z^1 \to 0$ and hence is in $\mathscr{M}$. This proves the claim and hence the lemma.

QED

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