THE DEFINITION OF A NEUTRAL TANNAKIAN CATEGORY

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Contents

1. Introduction

The goal of these notes is to present the definition of a neutral Tannakian category and introduce the vocabulary that frequently comes up in the same. We shall assume a minimal background in algebraic geometry and (homological) algebra, roughly at the level of first-year graduate courses.

We have extensively borrowed from the existing literature and make no claim about the originality of our expostion: in principal, we will closely follow section 1 of Lawrence Breen's paper on Tannakian categories (in the AMS *Motives* proceedings volume) as well as Jinhyun Park's online notes. We have no pretensions about precision here; instead general ideas have been emphasized over precise definitions and theorems.

The origin of Tannakian categories lies in representation theory. Let G be a locally compact abelian group and k a field of characteristic 0. Let $\widehat{G} = \operatorname{Hom}_k(G, k)$ be the set of characters of G. The classical Pontraygin duality states that $\hat{G}$ is locally compact and that there is some map $\rho : G \to \widehat{\widehat{G}}$ that is an isomorphism.

Can we do the something similar for affine group schemes? In other words, given the category of representations of affine group schemes, can we reconstruct the category of affine group schemes? (This is the goal of the Tannaka-Krein theorem.)

It turns out that the category of (finite-dimensional) representations of affine group schemes have a very rich structure, namely that of a neutral Tannakian category.

2. Affine group schemes and Hopf algebras

2.1. **Affine group schemes.** Let R be a commutative ring with identity.

Definition 1. [Category of Affine Schemes over R] Let $\mathbf{AffSch}_R$ be a category whose objects are $\mathrm{Spec}(S)$, where S is a finitely generated R-algebra, and morphisms are scheme morphisms $f : \mathrm{Spec}(S) \to \mathrm{Spec}(S')$ such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{Spec}(S) \arrow[rr, "f"] \arrow[dr] & & \mathrm{Spec}(S') \arrow[dl] \\ & \mathrm{Spec}(R) & \end{tikzcd}$$

Definition 2. [Affine Group Schemes] An *affine group scheme* is a group object in the category $\mathbf{AffSch}_R$.

Remark 1. In $\mathbf{AffSch}_R$, we have the following natural isomorphism

$$\mathrm{Spec}(S) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S') \xrightarrow{\sim} \mathrm{Spec}(S \otimes_R S')$$

If $G \in \mathbf{AffSch}_R$ is a group object, then additionally, we have the following scheme morphisms

$$\mu : G \times G \longrightarrow G$$
$$\iota : G \longrightarrow G$$

and $\mathrm{Spec}(R) \to G$. The terminology is obvious.

Example 1. [$GL_2(R)$] Take $R[a, b, c, d]$ and $f := ad - bc$. Then $GL_2(R) = \mathrm{Spec}(R[a, b, c, d])_f$, which is again same as $\mathrm{Spec}(R[a, b, c, d]) \setminus V(f)$, where $V(f)$ is a closed set in Zariski topology.

For the remainder of this section, we replace R by k, a field of arbitrary characteristic.

Theorem 1. *$\{$affine group schemes over $k\} \leftrightarrow \{$commutative Hopf algebras over $k\}^{op}$ is an antiequivalence of categories given by $A \longmapsto Spec(A)$, where A is a commutative Hopf algebra.*

2.2. **Hopf algebras.** An algebra A is a triple (A, μ, η) where A is a k-linear vector space,

$$\mu : A \otimes A \to A, \mu(a \otimes b) := ab$$

is the (k-linear) multiplication map and

$$\eta : k \to A,\ \mathrm{Im}(\eta) \subset Z(A),$$

is the unit map and such that the following diagrams commute:

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mathrm{id}\otimes\mu"] \arrow[d, "\mu\otimes\mathrm{id}"'] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"] & A \end{tikzcd}$$

and

$$\begin{tikzcd} A \otimes A \arrow[r, equal] & A \otimes A \arrow[r, equal] \arrow[d, "\mu"] & A \otimes A \\ A \otimes k \arrow[u, "\mathrm{id}\otimes\eta"] \arrow[r, "\sim"] & A & k \otimes A \arrow[l, "\sim"'] \arrow[u, "\eta\otimes\mathrm{id}"'] \end{tikzcd}$$

The idea behind a Hopf algebra is one 'dualizes' the above definition in the sense of introducing algebra morphisms $\Delta : A \to A \otimes A$ and $\epsilon : A \to k$ such that the following diagrams commute:

$$\begin{tikzcd} A \otimes A \otimes A & A \otimes A \arrow[l, "\mathrm{id}\otimes\Delta"'] \\ A \otimes A \arrow[u, "\Delta\otimes\mathrm{id}"] & A \arrow[l, "\Delta"'] \arrow[u, "\Delta"'] \end{tikzcd}$$

and

$$\begin{tikzcd} A \otimes A \arrow[r, equal] \arrow[d, "\mathrm{id}\otimes\epsilon"'] & A \otimes A \arrow[r, equal] & A \otimes A \arrow[d, "\epsilon\otimes\mathrm{id}"] \\ A \otimes k \arrow[r, "\sim"] & A \arrow[u, "\Delta"'] & k \otimes A \arrow[l, "\sim"'] \end{tikzcd}$$

Suppose $\exists S : A \to A$ such that the following diagram commute:

$$\begin{tikzcd} A \arrow[r, "\eta\circ\epsilon"] \arrow[d, "\Delta"'] & A \\ A \otimes A \arrow[r, "\mathrm{id}\otimes S{,}S\otimes\mathrm{id}"] & A \otimes A \arrow[u, "\mu"'] \end{tikzcd}$$

Definition 3. [Hopf algebra] The tuple $(A, \mu, \eta, \Delta, \epsilon, S)$ is said to be a *Hopf algebra*. The map $S : A \to A$ is called the *antipode* of A.

Since Spec is a contravariant functor, so corresponding to the morphism

$$\mathrm{Spec}(S) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S) = \mathrm{Spec}(S \otimes_R S) \to \mathrm{Spec}(S)$$

we have

$$\mu^* : S \longrightarrow S \otimes_R S$$

Remark 2. Recall that Spec is contravariant functor means that for any ring homomorphism $f : R \to S$, we get morphism of schemes $f^* : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$.

$$\begin{tikzcd} R \arrow[r] \arrow[d, "f"'] & \mathrm{Spec}(R) \\ S \arrow[r] & \mathrm{Spec}(S) \arrow[u, "f^*=\mathrm{Spec}(f)"'] \end{tikzcd}$$

Returning to the case of $GL_2(k)$, the coordinate ring $\mathrm{Spec}(k[a, b, c, d]_{ad-bc})$ can be viewed as Hopf algebra, via the following comultiplication:

$$\Delta(a) = 1 \otimes a + a \otimes 1,\ \Delta(b) = 1 \otimes b + b \otimes 1$$

$$\Delta(c) = 1 \otimes c + c \otimes 1,\ \Delta(d) = 1 \otimes d + d \otimes 1$$

Notice that we obtain a *commutative* Hopf algebra, which is good, since otherwise the antiequivalence of categories would not hold (no well defined notion of Spec in case of noncommutative algebras).

3. Representations of affine group schemes

Let k be a fixed field of arbitrary characteristic. Let G be an affine group scheme (over k) and V be a finite dimensional k-vector space.

Definition 4. [G-representation] A *representation* of G on V is a morphism

$$\rho : G \longrightarrow Aut_k(V)$$

Remark 3. We can also think of G-representation as morphism of affine schemes

$$G \times V \longrightarrow V$$

We will construct a category, namely, category of G-representations, where G is affine group scheme.

Definition 5. [$\mathbf{Rep}(G)$] Objects of category $\mathbf{Rep}(G)$ are representations (ρ, V) of G, where V is finite dimensional k-vector space. The morphisms between two G-representations (ρ_1, V_1) and (ρ_2, V_2) are maps $f : V_1 \to V_2$, such that for every $g \in G$, the following diagram commutes:

$$\begin{tikzcd} G \times V_1 \arrow[r, "\rho_1"] \arrow[d, "\mathrm{Id}_G \times f"'] & V_1 \arrow[d, "f"] \\ G \times V_2 \arrow[r, "\rho_2"] & V_2 \end{tikzcd}$$

Remark 4. A more general construction of $\mathbf{Rep}(G)$ over R, a commutative ring with identity, involves V being *projective R-module of finite rank*.

It turns out that in any case, $\mathbf{Rep}(G)$ has very rich structure as a category of k-vector spaces or R-modules.

4. Tensor categories and Tannakian categories

4.1. **Tensor categories.** Let $\mathcal{C} = \mathbf{Rep}(G)$. Then $\mathcal{C}$ has a product

$$\begin{array}{ccc} \mathcal{C} \times \mathcal{C} & \longrightarrow & \mathcal{C} \\ (\rho_1, \rho_2) & \longmapsto & \rho_1 \otimes \rho_2 \end{array}$$

which associates to every pair of G-representations (ρ_1, V_1) and (ρ_2, V_2), the representation

$$\rho_1 \otimes \rho_2 : G \longrightarrow Aut(V_1 \otimes V_2)$$

Abstractly, this product $\otimes$ satisfies the following functorial isomorphisms:

$$\text{Associativity } \rho_1 \otimes (\rho_2 \otimes \rho_3) \xrightarrow[\sim]{\varphi} (\rho_1 \otimes \rho_2) \otimes \rho_3$$

$$\text{Commutativity } \rho_1 \otimes \rho_2 \xrightarrow[\sim]{\psi} \rho_2 \otimes \rho_1$$

and existence of *unit*, which is the trivial G-representation

$$\mathbb{I} : G \longrightarrow Aut(\mathbf{1})$$

where $\mathbf{1}$ is one dimensional k vector space, with trivial G action. $\mathbb{I}$ is *the unit* object in $\mathcal{C}$. For every G-representation ρ

$$\mathbb{I} \otimes \rho \xrightarrow{\sim} \rho \xrightarrow{\sim} \rho \otimes \mathbb{I}$$

Definition 6. [Tensor category] The system $(\mathcal{C}, \otimes, \mathbb{I})$ is an example of *monoidal/tensor category*. A functor between two tensor categories $\mathcal{C}_1$ and $\mathcal{C}_2$ is a functor that satisfies the following condition:

$$F(A) \otimes F(B) \xrightarrow{\sim} F(A \otimes B)$$

F is assumed to be compatible with associativity and commutativity conditions (coherence conditions of Maclane) of $\otimes$. Such functors are called *tensor functors*.

Saavedra Rivano calls $(\mathcal{C}, \otimes, \mathbb{I})$ a *ACU* $\otimes$ category. A (trivial) example of a tensor category: let R be a commutative ring with identity and let $\mathbf{Mod}(R)$ be category of R-modules. This is a tensor category with $\mathbb{I} = R$.

Definition 7. [Invertible objects] Let $\mathcal{C}$ be a tensor category. An object $L \in \mathrm{Obj}(\mathcal{C})$ is *invertible* if there exists a unique object $L^{-1} \in \mathrm{Obj}(\mathcal{C})$ and a morphism

$$L \otimes L^{-1} \xrightarrow[\sim]{inv} \mathbb{I}$$

Remark 5. Since $L \otimes L^{-1} \xrightarrow{\sim} L^{-1} \otimes L$, we only need to specify one side.

Remark 6. [Exercise] In $\mathbf{Mod}(R)$, an object is invertible if and only if it is projective of rank 1.

Definition 8. [Internal hom] Let $(\mathcal{C}, \otimes)$ be a tensor category. Let $X, Y \in \mathrm{Obj}(\mathcal{C})$. Consider the functor:

$$\begin{array}{rcccl} F & : & \mathcal{C} & \longrightarrow & \mathbf{Sets} \\ & & T & \longmapsto & \mathrm{Hom}(T \otimes X, Y) \end{array}$$

If $F = \mathrm{Hom}(\_, K)$, where K is some object in $\mathcal{C}$, then define $\underline{\mathrm{Hom}}(X, Y) := K$. More explicitly, we have the following:

$$\mathrm{Hom}(T \otimes X, Y) = \mathrm{Hom}(T, \underline{\mathrm{Hom}}(X, Y))$$

Remark 7. The above definition is saying that the functor F is *representable*. It is important to remember that $\underline{\mathrm{Hom}}(X, Y)$ is an object of $\mathcal{C}$.

Example 2. In $\mathbf{Mod}(R)$, we can see that $\underline{\mathrm{Hom}}(X, Y) = \mathrm{Hom}_R(X, Y)$, where X, Y are R-modules. To prove this, we can unwind the definitions as follows:

$$\begin{array}{rcccl} F & : & \mathbf{Mod}(R) & \longrightarrow & \mathbf{Sets} \\ & & T & \longmapsto & \mathrm{Hom}(T \otimes X, Y) \end{array}$$

By Hom-tensor adjointness, we know that $\mathrm{Hom}_R(T \otimes X, Y) = \mathrm{Hom}_R(T, \mathrm{Hom}_R(X, Y))$, and hence $F = \mathrm{Hom}_R(\_, \mathrm{Hom}_R(X, Y))$, which by definition proves that $\underline{\mathrm{Hom}}(X, Y) = \mathrm{Hom}_R(X, Y)$.

Definition 9. Let $\mathrm{ev}_{X,Y} : \underline{\mathrm{Hom}}(X, Y) \otimes X \to Y$ be the morphism corresponding to $Id_{\underline{\mathrm{Hom}}(X,Y)}$.

Remark 8. By definition, $\mathrm{Hom}(\underline{\mathrm{Hom}}(X, Y) \otimes X, Y) = \mathrm{Hom}(\underline{\mathrm{Hom}}(X, Y), \underline{\mathrm{Hom}}(X, Y))$ (take $T = \underline{\mathrm{Hom}}(X, Y)$ to see this). So the correspondence in the above definition makes sense.

Definition 10. [Dual] The dual of X is defined as $\hat{X} := \underline{\mathrm{Hom}}(X, \mathbb{I})$.

Example 3. In $\mathbf{Mod}(R)$

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(X,Y)\otimes X \arrow[r, "\mathrm{ev}_{X,Y}"] \arrow[d, equal] & Y \arrow[d, equal] \\
\mathrm{Hom}_R(X,Y)\otimes X \arrow[r, "\mathrm{ev}_{X,Y}"] & Y
\end{tikzcd}$$

is given by $f \otimes x \longmapsto f(x)$, so the notion of evaluation with internal hom coincide with our usual notion of evaluation maps on R-modules.

Definition 11. [Reflexive objects] Suppose we are given the following diagram:

$$\begin{tikzcd}
 & \underline{\mathrm{Hom}}(X,\mathbb{I}) \arrow[dr, "\mathrm{ev}_{X,\mathbb{I}}"] & \\
X\otimes \hat{X} \arrow[ur, "\psi"] \arrow[rr, "\mathrm{ev}_{X,\mathbb{I}}\circ\psi"'] & & \mathbb{I}
\end{tikzcd}$$

where X is an object in some tensor category. If $\mathrm{ev}_{X,\mathbb{I}} \circ \psi$ is an isomorphism, we say that X is *reflexive*. Equivalently, if all objects of a given tensor category are reflexive, we call the category reflexive.

Remark 9. We note that the above definition does correspond to our usual notion of reflexiveness: from the definition of internal hom:

$$\mathrm{Hom}(X\otimes\hat{X},\mathbb{I}) = \mathrm{Hom}(X,\underline{\mathrm{Hom}}(\hat{X},\mathbb{I}))$$

If $\mathrm{ev}_{X,\mathbb{I}} \circ \psi$ is an isomorphism, we get a correspondence

$$\mathrm{ev}_{X,\mathbb{I}}\circ\psi \qquad \leftrightarrow \qquad X \xrightarrow{\quad\sim\quad} \hat{\hat{X}}$$

with $\mathrm{ev}_{X,\mathbb{I}} \circ \psi \in \mathrm{Hom}(X\otimes\hat{X},\mathbb{I})$ and $X \xrightarrow{\sim} \hat{\hat{X}} \in \mathrm{Hom}(X,\underline{\mathrm{Hom}}(\hat{X},\mathbb{I}))$.

Obviously, not all tensor categories are reflexive. For example, $\mathbf{Mod}(R)$ is not reflexive for the following reason. Take $R=\mathbb{Z}$, then $\mathbf{Mod}(R)$ is same as category of abelian groups ($\mathbf{AbGrp}$). For $\mathbb{Z}/2\mathbb{Z} \in \mathbf{AbGrp}$, we have $\widehat{\mathbb{Z}/2\mathbb{Z}} = \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}) = 0$. This implies, $\widehat{\widehat{\mathbb{Z}/2\mathbb{Z}}} = \hat{0} = 0 \neq \mathbb{Z}/2\mathbb{Z}$.

4.2. **Neutral Tannakian categories.**

Definition 12. [Rigid tensor category] A tensor category $\mathcal{C}$ is *rigid* if

(1) $\underline{\mathrm{Hom}}(X,Y)$ exists for every $X,Y \in \mathrm{Obj}(\mathcal{C})$.
(2) All natural maps

$$\bigotimes_{i\in I}\underline{\mathrm{Hom}}(X_i,Y_i) \to \underline{\mathrm{Hom}}\left(\bigotimes_{i\in I}X_i, \bigotimes_{i\in I}Y_i\right)$$

are isomorphisms.
(3) $(\mathcal{C},\otimes)$ is a reflexive category.

Before stating the last and most important definition of these notes, we note some definitions about functors.

Definition 13. [Exact functor] A functor F from an *abelian category* $\mathbf{Ab}_1$ to some other abelian category $\mathbf{Ab}_2$ is said to be *exact* if for every exact sequence in $\mathbf{Ab}_1$:

$$0 \to A_1 \to A_2 \to A_3 \to 0$$

we obtain an exact sequence in $\mathbf{Ab}_2$:

$$0 \to F(A_1) \to F(A_2) \to F(A_3) \to 0$$

Definition 14. [Faithful functor] A functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ is said to be faithful if for $X, Y \in \mathrm{Obj}(\mathcal{C}_1)$, we have an isomorphism in $\mathcal{C}_2$, $F(X) \xrightarrow[\sim]{g} F(Y)$, then there exists an isomorphism in $\mathcal{C}_1$, $X \xrightarrow[\sim]{f} Y$, such that $g = F(f)$.

Finally, with all this vocabulary, we have:

Definition 15. [Neutral Tannakian category] Let k be a field of arbitrary characteristic. A *neutral Tannakian category* over k is a rigid abelian tensor category $\mathcal{C}$ with a (k-linear) exact faithful tensor functor $\omega : \mathcal{C} \to \mathrm{Vect}_k$, called the *fiber functor*.

Remark 10. As Breen notes, the idea behind ω is that of a "vector bundle" over the category $\mathcal{C}$ whose fiber over an object $X \in \mathrm{Obj}(\mathcal{C})$ is the k-vector space $\omega(X)$.

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