SUMMARY : DOMAINS OF HOLOMORPHY
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24TH SEPTEMBER 2012

We begin by exploring one of the major differences in domains of functions of single variable and several complex variables. In single variable case we have the following situation :

Proposition 0.1. *Let U be a domain (open connected set) of $\mathbb{C}$. Then there exist f holomorphic on U, $f \in H(U)$ which cannot be analytically extended to any larger open set.*

It is important to note that the above result is not true in $\mathbb{C}^n$ where $n \geq 2$. So we can find some domain U and V such that $U \subset V \subset \mathbb{C}^n$ such that functions holomorphic on U is also holomorphic on V.

Proposition 0.2. *Let $\Delta(0,r)$ be an open polydisc in $\mathbb{C}^n$ with $n > 1$. Set $r = (r_1, \ldots, r_n)$ and $r^{'} = (r_1, \ldots, r_{n-1})$. Let U be an connected open subset of $\Delta(0,r)$ and for each $z^{'} \in \Delta(0, r^{'})$ set $U_{z'} = \{z_n \in \mathbb{C} : (z^{'}, z_n) \in U\}$. Assume that U has the following properties :*

(i) there is a fixed $s < r_n$ such that $\bar{\Delta}(0,s)$ contains the complement of $U_{z'}$ in $\Delta(0, r_n)$ for each $z^{'} \in \Delta(0, r^{'})$;
(ii) the equality $U_{z'} = \Delta(0, r_n)$ holds for all $z^{'}$ in some open subset of $\Delta(0, r^{'})$.

Then every holomorphic function on U has a holomorphic extension to $\Delta(0,r)$.

Proof. See Section 2.5 in [1]. □

It is easy to construct examples of the above situation :

Example : Let $\Delta(0,r)$ be an open polydisc and $U = \Delta(0,r) - K$, where K is any compact subset of $\Delta(0,r)$. which does not separate $\Delta(0,r)$. Clearly U and $\Delta(0,r)$ satisfies the conditions of above proposition. Thus any function holomorphic on $\Delta(0,r) - K$ extends to be holomorphic on all of $\Delta(0,r)$.
Example: Let A be the open annulus $\Delta(0,1) - \bar{\Delta}(0, 1/2)$ and set $U = (\Delta(0,1) \times A) \cup (\Delta(0,1/2) \times \Delta(0,1)) \subset \Delta(0,(1,1))$. Here U and $\Delta(0,(1,1))$ both satisfies the conditions of above proposition. The result is a solid cylinder of length 1 and radius 1 with hole of radius 1/2 drilled half way through from one end.

Definition. An open set $U \subset \mathbb{C}^n$ is called a domain of holomorphy if there exist $f \in H(U)$ such that for all z on the boundary of U and each polyradius r, there is no holomorphic function on $\Delta(z,r)$ which is equal to f on a component of $\Delta(z,r) \cap U$.
In other words, U is a domain of holomorphy if there is a holomorphic function on U which has no local holomorphic extension across part of the boundary of U.

Definition. Holomorphic convex hull of a compact set K in U denoted by $C(K|U) = \{z \in U : \forall f \in H(U),\ |f(z)| \leq \|f\|_K\}$ where $\|f\|_K = \sup\{|f(z)| : z \in K\}$ is the operator seminorm of K.

Moreover an open set $U \subset \mathbb{C}^n$ is said to be holomorphically convex if $C(K|U)$ is compact for each compact subset $K \subset U$.

Proposition 0.3. *If U is an open set in $\mathbb{C}^n$, then U is a domain of holomorphy if and only if U is holomorphically convex.*

Proof. See Section 2.5 in [1]. □

References

[1] Joseph L. Taylor, *Several Complex Variables with connections to Algebraic Geometry and Lie Groups* Volume 46 AMS, Providence, Rhode Island.