Summary: Morse Theory

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1 Motivation

We shall consider a generating function, $F(\underline{z}) \in O(\mathcal{D})$, where $\mathcal{D} \in \mathbb{C}^d$. We know that $a_{\underline{n}}$ can be evaluated by the Cauchy integral formula

$$a_{\underline{n}} = \frac{1}{2\pi i}^d \int_\Gamma \frac{1}{z^{\underline{n}}} F(\underline{z}) d\underline{z}. \tag{1}$$

Denote $\omega = \frac{1}{z^{\underline{n}}} F(\underline{z}) d\underline{z}$ and call it a *differential form.*

Now observe that Γ is a d-dimensional torus; in other words, Γ is the Cartesian product of d circles, $\mathbb{S}^1$.

Denote by ∂, the boundary operator and note that $\partial \mathbb{S}^1 = 0$, so $\partial\Gamma = 0$. Therefore $\Gamma \in \ker(\partial)$ and is called a *cycle.*

Denote by $d : \Omega^k(\mathcal{D}) \to \Omega^{k+1}(\mathcal{D})$ the differential operator. We can see that the form ω is *closed*, that is $\omega \in \ker(d)$, via the computation of $d\omega$.

It can be shown via computation that deforming Γ into $\Gamma + \partial M$ and translating ω into $\omega + d\omega_0$ both do not change the value of the integral (1). We simply need to expand the sum and apply Stokes's theorem.

We shall investigate how to easily compute the integral (1) by deformations of Γ. In particular we would like to use Morse theory to characterize $\mathcal{H}_*(\mathcal{D}) = {}^{\ker(\partial)}/_{\operatorname{im}(\partial)}$.

intro to defs.
Definition 1.1. Let X, Y be topological spaces and

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

be continuous maps. We say that f is *homotopic* to g, $f \sim g$, if there is a continuous map $F : X \times [0, 1] \to Y$ such that for all $x \in X$, $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$.
Definition 1.2. Let x, Y be topological spaces. We say that X and Y are *homotopy equivalent* if there are

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

continuous maps such that $f \circ g$ is homotopic to id_Y and $g \circ f$ is homotopic to id_X.

Though homeomorphic topological spaces are homotopy equivalent, it is important to note that not all homotopy equivalent topological spaces are homeomorphic.

Example 1. Let $X = \mathbb{D}^3$, the continuous sphere in $\mathbb{R}^3$ and $Y = \{q\}$, for $q \in \mathbb{R}^3$.

These topological spaces are clearly not homeomorphic; however we shall see that they are homotopy equivalent.

Define $f : X \to Y, p \to q$ and $g : Y \to X, q \to 0$.

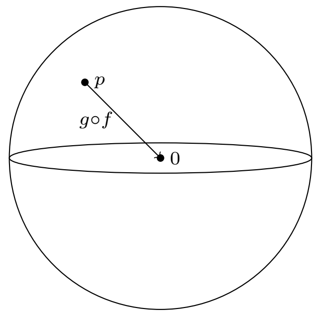

Figure 1: The map $g \circ f$.

Then $(f \circ g)(q) = q$, so $f \circ g = \mathrm{id}_y$ and $(g \circ f)(p) = 0$. $g \circ f$ does not equal id_X but they are homotopic via the map

$$\begin{array}{rccc} F: & X \times [0,1] & \to & X \\ & (x,t) & \to & tx. \end{array}$$

Theorem 1.3 (Homotopy principle)**.** *Let X, Y be topological spaces. If X is homotopy equivalent to Y then $H_*(X) \cong H_*(Y)$.*

2 Morse theory of closed surfaces

In this section X is an orientable surface in $\mathbb{R}^3$.

Definition 2.1. A *Morse function* in X is a smooth function $h : X \to \mathbb{R}$ such that h has only nondegenerate forms.

Definition 2.2. A point $p \in X$ is a *critical point* of a Morse function H if $dh|_p = 0$.

Definition 2.3. The *Hessian* of h at p is the matrix

$$H(h,p) = \begin{pmatrix} \partial_x^2 h|_p & \partial_x \partial_y h|_p \\ \partial_y \partial_x h|_p & \partial_y^2 h|_p \end{pmatrix}$$

where (x, y) are local coordinates of a neighbourhood of p.

A critical point is non-degenerate if $\det H(h,p) \neq 0$.

Example 2. The projection onto the z-axis is not a Morse function for the surface in Figure 2 because its critical points are degenerate. This is due to the geometry of the surface.

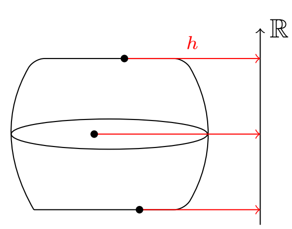

Figure 2: $h = \mathrm{pr}_z$ is not a Morse function for this surface.

For the sphere (see Figure 3), the projection onto the z-axis is a Morse function because its critical points are a maximum and a minimum. However the resultant function $h'(p) = m$ for all p is not a Morse function of the sphere because every point is a degenerate critical point. This is due to the definition of the function.

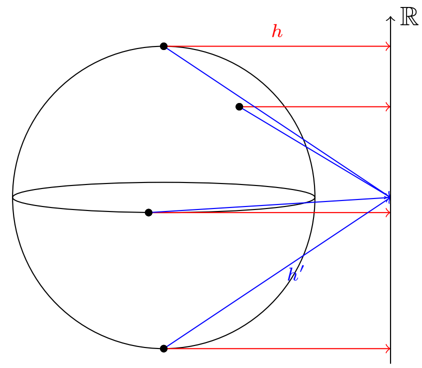

Figure 3: $h = \mathrm{pr}_z$ is a Morse function for the sphere, but h' which sends everything to 0 is not.

Theorem 2.4. *Let x be a smooth closed surface and $h : X \to \mathbb{R}$ be a smooth function with exactly 2 critical points. Then X is homeomorphic to the sphere.*

Proof. Start by examining the neighbourhoods of the 2 critical points and then glue them together.

In the text they prove the more general case of k critical points in Theorem B.17. □

Define $X^{[a,b]} = \{p \in X : a \leq h(p) \leq b\}$, $X^a = \{p \in X : a \leq h(p)\}$ and $\mathcal{L}_a(h) = \{p \in X : h(p) = a\}$.
Lemma 2.5. *If $h : X \to \mathbb{R}$ is a Morse function without any critical points in $[a, b]$ then $X^{[a,b]} \cong \mathcal{L}_a(h)$.*
Lemma 2.6. *If $h : X \to \mathbb{R}$ is a Morse function on a closed surface it must have finitely many critical points.*

These lemmas lead to the Morse Lemma.
Lemma 2.7 (Morse lemma)**.** *If $h : X \to \mathbb{R}$ is a Morse function and h has no critical values in $[a, b]$ then X^a is homotopy equivalent to X^b.*

Proof. Use the previous lemmas to show that $X[a-\epsilon, a]$ and $X[a-\epsilon, b]$ are both homeomorphic to $\mathcal{L}_{a-\epsilon}(h) \times [0, 1]$ for ϵ small enough. □

Example 3. In this example we shall do a handlebody decomposition of the torus.

If we take our Morse function, $h = \mathrm{pr}_z$ then h has 4 critical points, p_-, p_1, p_2 and p_+ as seen in Figure 4. Let $m_- = h(p_-)$, $m_1 = h(p_1)$, $m_2 = h(p_2)$ and $m_+ = h(p_+)$.

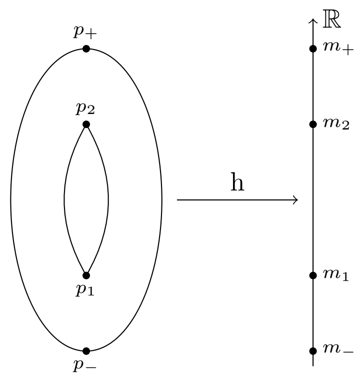

Figure 4: Torus with the critical points of h marked.

Around the minimum the torus $h(p) = x^2 + y^2 + m_-$. So $X^{m_1+\epsilon} \cong \mathbb{D}^2$. See Figure 5.

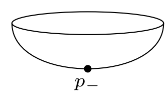

Figure 5: Near the minimum of the torus.

Around p_1, $h(p) = -u^2 + v^2 + m_1$. The shape is a saddle and is isomorphic to $\mathbb{D}^1 \times \mathbb{D}^1$. See Figure 6.

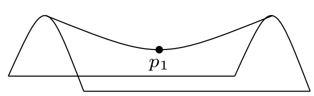

Figure 6: Near p_1.

We can attach the saddle to the minimum's sphere using a union via the gluing function g_1. See Figure 7. So $X^{m_1+\epsilon} \cong \mathbb{D}^2 \cup_{g_1} (\mathbb{D}^1 \times \mathbb{D}^1)$.

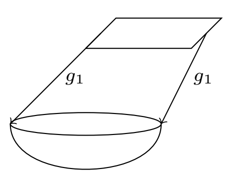

Figure 7: Gluing g_1.

For p_2 we glue using the function g_2 giving $X^{m_2+\epsilon} \cong \left(\mathbb{D}^2 \cup_{g_1} (\mathbb{D}^1 \times \mathbb{D}^1)\right) \cup_{g_2} (\mathbb{D}^1 \times \mathbb{D}^1)$. See Figure 8.

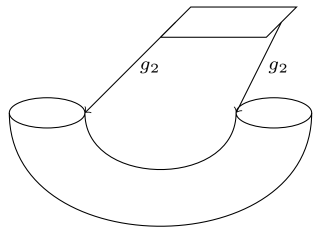

Figure 8: Gluing g_2.

Finally, we glue the maximum's sphere with g_1 to get $X \cong \Big(\big(\mathbb{D}^2 \cup_{g_1} (\mathbb{D}^1 \times \mathbb{D}^1)\big) \cup_{g_2} (\mathbb{D}^1 \times \mathbb{D}^1)\Big) \cup_{g_1} \mathbb{D}^2$.

Definition 2.8. A *cell complex* is a topological space defined as follows:

$$
\begin{aligned}
X[0] &= e_1^{\text{-}0} \sqcup e_2^{\text{-}0} \sqcup \cdots \sqcup e_i^{\text{-}0} \\
X[j] &= (e_1^{\text{-}j} \sqcup e_2^{\text{-}j} \sqcup \cdots \sqcup e_i^{\text{-}j}) \cup_g X[j-1] & \forall j \geq 1
\end{aligned}
$$

The decomposition of the torus in Example 3 is a cell complex.