Math 303 - paradoxes in set theory - Lecture 1

1 Introduction

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You:

This course is specifically designed for math minors and other people with a casual interest.

The course: Books Halmos Naive set theory

Cohen Set theory and the continuum hypothesis

Old fashioned, but cheap and classic.

Other resources online wikipedia, wikibooks

The joy of sets by Keith Devlin

The Aha books by Martin Gardner

now reprinted by the MAA

http://math.sfu.ca/~kyeats/teaching/math303.html

Office hours

Wed afternoon 2:30

Monday after class

Grading

Homework 25%

Midterm 25%

Final 50%

✓

What should I do with the notes? blanks

Homework box? NO

(2) What is a set? (Halmos section 1 (see also Cohen II.1))

stuff with something in common

a group of somethings

each trial from an experiment.

integers $\mathbb{Z}$

even integers $\{2, 4, 6, 8, \ldots\}$

{the moon, George Washington, freedom} ← American book

Idea: A collection of objects can be viewed as an object in its own right, namely a set

Google ~~sets http://labs.google.com/sets~~ gone Sept 5

Key is the idea of belonging to, or being a member of, or element of a set.

we write

$$x \in A$$ to say x is an element of the set A

we write

$$x \notin A$$ to say x is not an element of the set A

eg If A is the set of even numbers then

$$2 \in A \qquad 3 \notin A$$

eg $\{1,2\} \in \{\{1\}, \{1,2\}, \{1,2,3\}\}$

Notation note: Halmos tells us ϵ is the Greek letter epsilon.
In modern notation it has evolved into a symbol in its own right $\in$

http://en.wikipedia.org/wiki/Element_%28mathematics%29#Notation_and_terminology

Halmos uses ϵ' instead of $\notin$ for not in

When are two sets equal?

that is, what should

$A=B$ mean for A and B sets?

They contain the same elements ~~in the same order~~ ← list

$A \subseteq B$ and $B \subseteq A$ implies $A = B$

every element of A is also an element of B

and B has no extra elements

$\boxed{\text{Axiom of extension} \quad \text{two sets are equal if and only if they have the same elements}}$

Note ① if and only if : P if and only if Q

mean P is logically equiv to Q

ie if P then Q

if Q then P

② Aren't I going to write this in logical symbols?

We will but not right now.

③ The axiom isn't vacuous – it says something nontrivial about what belonging means

An example from Halmos:

$a \in A$

Suppose A and B are people and $\in$ means is an ancestor of

Then the extension axiom would mean two people are the same iff they have the same ancestors

short hand for iff

for all a ancestor of A we have a ancestor of B and vice versa

Not true because of siblings

④ Our goal for the next few weeks is to set up axioms for sets. Then we'll really know what sets are.

③ Subsets

If A and B are sets
and every element of A is an element of B

then A is a subset of B
written $A \subseteq B$

eg even integers is a subset of all integers

eg $A \subseteq A$

Notation note Halmos uses the older tradition and writes
$A \subset B$ even if A and B are equal
I will compromise $A \subseteq B$ for A subset of B
$A \subsetneq B$ for subset but not equal.

You can do whatever you like

Properties Let A, B, and C be sets

1. $A \subseteq A$ reflexivity
2. if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$ transitivity
3. if $A \subseteq B$ and $B \subseteq A$ then by the axiom of extension $A = B$

Note that $\in$ and $\subseteq$ are very different

eg $1 \in \{1, 2, 3\}$ but $\{1, 2\} \notin \{1, 2, 3\}$

$\{1, 2\} \in \{\boxed{\{1, 2\}}, 1, 487, 2\}$

and $\{1, 2\} \subseteq \{\{1, 2\}, \underline{1}, 487, \underline{2}\}$

Another example it is always true that

$A \subseteq A$

but is it ever true that $A \in A$?

For the break

Read sections 1 and 2 of Halmos

Is it possible to have $A \in A$?

Meet your neighbor

?

④ How to specify sets

In everyday life how do we specify a set

we give a property that the elements must satisfy

eg the set of $\underbrace{\text{even} \ \boxed{\text{integers}}}$

property: even

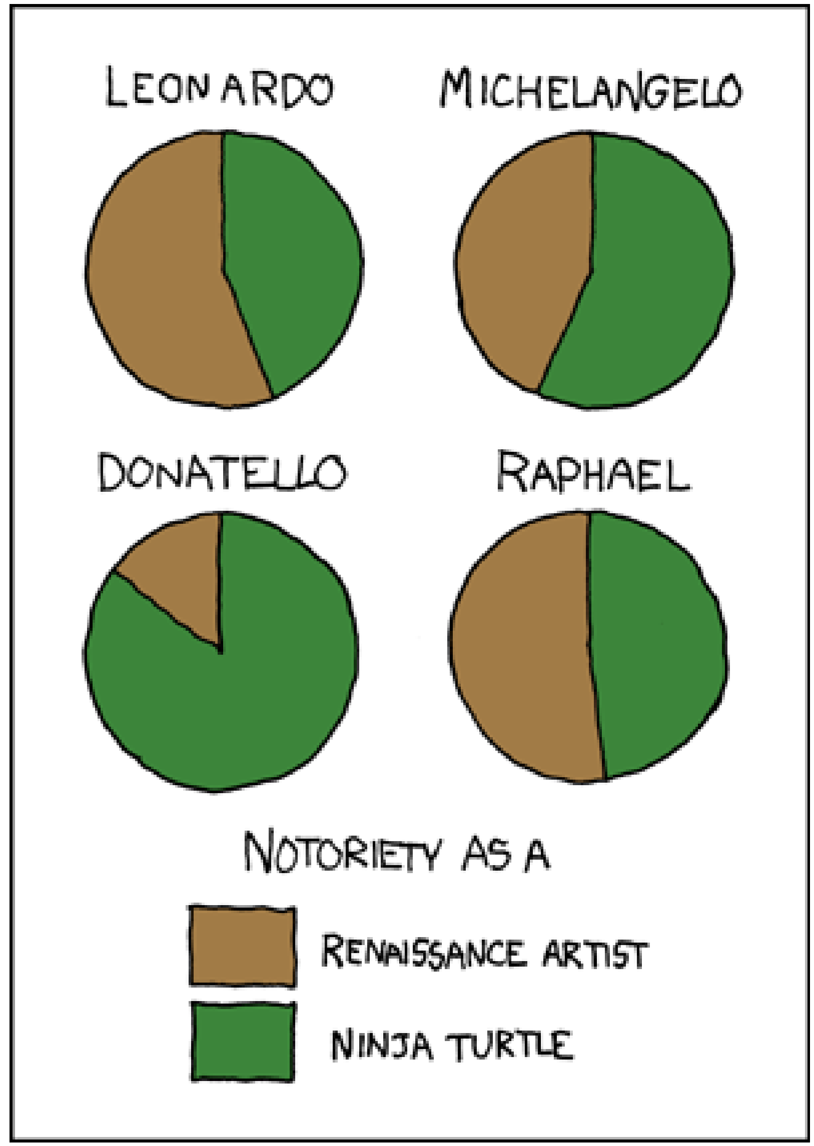
bigger set: integers

eg the set of conifers $\boxed{\text{trees}}$

Some of the funny results google sets gave were from guessing a different property

An example from xkcd

http://xkcd.com/197/



However, there's a problem with this naive notion.

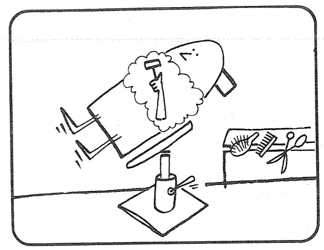
A serious problem.

Russell's paradox

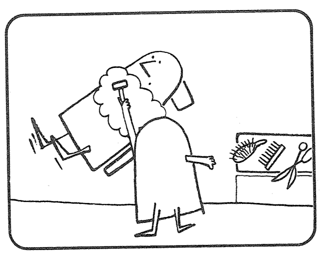
our first paradox

The Barber Paradox

The famous barber paradox was proposed by Bertrand Russell. If a barber has the sign at the left in his window, who shaves the barber?



If he shaves himself, then he belongs to the set of men who shave themselves. But his sign says he *never* shaves anyone in this set. Therefore he *cannot* shave himself.



If someone else shaves the barber, then he's a man who doesn't shave himself. But his sign says that he *does* shave *all* such men. Therefore no one else can shave the barber. It seems as if *nobody* can shave the barber!

what if the barber is a woman.

(from Aha! Gotcha by Martin Gardner)

what we'll do

Bertrand Russell proposed the barber paradox to dramatize a famous paradox he had discovered about sets. Some constructions seem to lead to sets that should be members of themselves. For example, the set of all things that are not apples could not be an apple, so it must be a member of itself. Consider now the set of all sets that are *not* members of themselves. Is it a member of itself? However you answer, you are sure to contradict yourself.

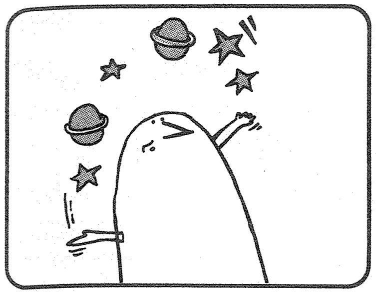
One of the most dramatic turning points in the history of logic involves this paradox. Gottlob Frege, an eminent German logician, had completed the second volume of his continuing life's work, *The Fundamentals of Arithmetic,* in which he had thought he had developed a consistent theory of sets that would serve as the foundation of all mathematics. The volume was at the printer's when Frege received a letter from Russell, in 1902, telling him about the paradox. Frege's set theory permitted the formation of the set of all sets not members of themselves. As Russell's letter made clear, this apparently well-formed set is self-contradictory. Frege had time only to insert a brief appendix that begins: "A scientist can hardly encounter anything more undesirable than to have the foundation collapse just as the work is finished. I was put in this position by a letter from Mr. Bertrand Russell. . . ."

It has been said that Frege's use of the word "undesirable" is the greatest understatement in the history of mathematics.

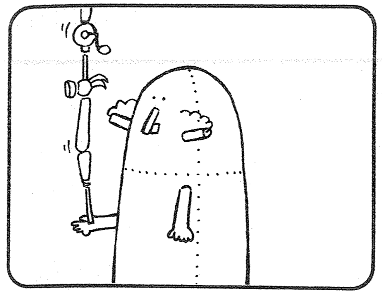
We will explore a few more paradoxes of this type and mention various approaches to eliminating them. One way out of this dilemma is to decide that the description "the set of all sets that do not contain themselves" does not name a set. A more sweeping and radical solution would be to insist that set theory allow no sets that are members of themselves.

Astrologer, Robot, and Catalog

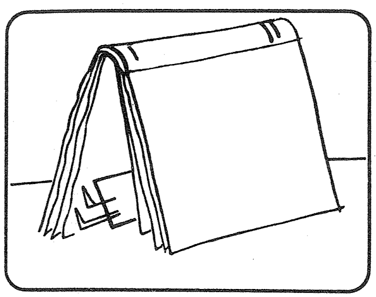
more variations



How about the astrologer who gives advice to all astrologers, and only those, who do not advise themselves? Who advises the astrologer?



Or the robot who repairs all robots who do not repair themselves? Who repairs the robot?



Or a catalog that lists all catalogs that do not list themselves? What catalog lists this catalog?

These are all variations of Russell's paradox. In each case the proposed definition for a set, S, is that it contain all those objects and only those objects that do not stand in a certain relation, R, to themselves. If one asks whether or not S belongs to itself, the paradox becomes apparent. Here are three classical variations on this theme.

1. Grelling's paradox is named for its discoverer, the German mathematician Kurt Grelling. We divide all adjectives into two sets: self-descriptive and non-self-descriptive. Words such as *English, short,* and *polysyllabic* are self-descriptive. Words such as *German, monosyllabic,* and *long* are non-self-descriptive. Now we ask: To which class belongs the adjective *non-self-descriptive?*

2. Berry's paradox gets its name from G. G. Berry, an Oxford University librarian who communicated it to Russell. It concerns "the smallest integer that cannot be expressed in less than thirteen words." Since this expression has 12 words, to which set does the integer it describes belong: the set of integers that can be expressed in English with less than 13 words, or the set of integers that can be expressed only with 13 words or more? Either answer leads to a contradiction.

3. The philosopher Max Black expressed the Berry paradox in a fashion similar to the following version: Various integers are mentioned in this book. Fix your attention on the smallest integer that is not referred to in any way in the book. Is there such an integer?

Suppose there was a Wikipedia page that was a category page for Category pages that don't list themselves

Category – Insects of the Amazon
– Tarantula

Category – Pages created in 2011
– someone's favorite webcomic (2011)

a category page which does not list itself

say I create this category page today. Then this page list itself

- Categor Pgs created in 2011

Category - Category pages that do not list themselves

- Insects of the Amazon

(Pages created in 2011 is not listed here)

Question? Should "category pages that do not list themselves" be listed in category pages that do not list themselves?

How do we state Russell's paradox in set theory?

let B be the set of all sets C with $C \notin C$

so B is the set of all sets which do not have themselves as a member

$C \notin C$ seems almost vacuous

But Is B in itself?

if $B \notin B$ so then B is a set of the form C with $C \notin C$
so $B \in B$ contradiction

if $B \in B$ then B has the property $B \notin B$ contradiction.

So we get a contradiction either way. That is the paradox

How do we fix this? What is wrong with the specification of B?

Axiom of Specification or Subset Selection

For every set A and every condition $S(x)$ there is a set B consisting of exactly the elements of A for which $S(x)$ holds

ie $B = \{ x \in A \mid S(x) \text{ is true} \}$

Notes ① eg a condition might be "x is even"

$B = \{ x \in \mathbb{Z} \mid x \text{ is even} \}$

another condition "x is a set with 2 elements"

let $A = \{ \{1\}, \{1,2\}, \{1,2,3\}, \ldots \}$ $B = \{ x \in A \mid x \text{ has 2 elements} \}$

$= \{ \{1,2\} \}$

② I haven't made this rigorous yet because I haven't said what properties are allowed

Ultimately want $S(x)$ to be a sentence in first order logic.

for now $S(x)$ is anything built of

$\in$, $=$, and, or, not, implies,

iff, there exists, for all

③ How does this resolve Russell's paradox

The property $S(C)$ is $C \notin C$

What's A? It should be the set of all sets.

So to resolve Russell's paradox we conclude there is no set of all sets

we have no universe of discourse

④ Things like the set of all sets are called proper classes. They are too big to be sets themselves.

infinitely

⑤ Next time

More ways to build sets

$\boxed{\text{Please read sections 3 and 4 of Halmos.}}$

look at website for homework.