Math 303, Fall 2011, Lecture 12

① Free and bound variables

Are the following well formed? (don't worry about parentheses as long as its clear)

$x=y$ yes good

$\forall x \exists y \exists z (y=z)$ yes not good

$\exists x (y \in z)$ yes not good

$\exists x \forall x (x \in y)$ yes not good

x appears in $\forall x (x \in y)$ but it isn't free

$\forall y (\boxed{\exists x (x \in y)} \wedge \boxed{\exists x (y \in x)})$ yes

not good because share non-free variables to describe what is unpleasant about these we need a definition

Definition Each $\boxed{\text{occurence}}$ of a variable symbol in a wff is free or bound as described below

(A) Every variable appearing in a formula built using rules (1) or (2) is free

(B) An occurence of a variable in a formula coming from rule (3) is free or bound according to whether it is free or bound in A or B

(Note we're talking about a specific occurence of the variable and in particular this occurrence is only in one of A or B)

(C) The free and bound occurrences of a variable in $\exists x A$ or $\forall x A$ are the same as in A except that every occurrence of x $\boxed{\text{which is free}}$ in A is now bound. As is the x in $\exists x$ and $\forall x$

eg $x \in y$

free good

eg $\forall x (x \in y)$

bound good

eg $(x \in y) \wedge (\forall x (x \in y))$

not good

note some xs are free and some bound in the same formula

x is free in but not in

eg $\exists x ((x \in y) \wedge (\forall x (x \in y)))$

not good in building it we used the previous one which was not good

eg $\forall y \exists x ((x \in y) \wedge (\forall x (x \in y)))$

not good ''

But there is something very unpleasant going on even though these are well formed

Definition A wff is $\boxed{\text{good}}$ if every time 3 was applied
The only variables common to A and B are free in all of their occurrences
and every time 4 was applied x appeared only as a free variable in A, and did so appear.

Note Good formulas are much easier for a person to understand. Generally best to use good formulas, you and your reader are less likely to get confused.
However all our rules are ok for non-good formulas too.

The only tricky question is what do

$\forall x\, A$

$\exists x\, A$ mean if x doesn't appear in A?

Answer the $\forall x$ or $\exists x$ has no meaning
The formula has the same meaning as just A

eg Which of the formulas given so far today are good

If you stick to good formulas you are less likely to get confused.

Definition A formula with no free variables is a sentence or statement

eg $\forall x \exists y (x \in y)$ sentence true in set theory

eg $\forall x \exists y (y \in x)$ sentence false in set theory
$\emptyset$

eg $\exists y (x \in y)$ not a sentence no meaning unless x is given a value.

Note a sentence is something which can be true or false

Try the above examples

But be careful we're very used to saying things like

$\boxed{x+y=y+x}$ for $x, y \in \omega$

but we are quantifying here

This corresponds to the sentence

$$\forall x \forall y \left((x+y=y+x) \wedge (x \in \omega) \wedge (y \in \omega) \right)$$

② Some important abbreviations

$x \subseteq y$ abbreviates $\forall z\,((z \in x) \to (z \in y))$

$x = \{y\}$ abbreviates $\underbrace{(y \in x)}_{\text{y is in x}} \wedge \underbrace{\forall z((z \in x) \to (z = y))}_{\text{and nothing else is in x}}$

You try the next 3 for the break

$\boxed{x = \{y, z\}}$ abbreviates $(y \in x) \wedge (z \in x) \wedge \forall a\,((a \in x) \to ((a = y) \vee (a = z)))$

$x = \bigcup y$ abbreviates $\forall z\, \forall y'\, (((z \in y') \wedge (y' \in y)) \to (z \in x))$

$x = (y, z)$ abbreviates $(\forall a'(a' = \{y\} \to a' \in x)) \wedge \forall a((a \in x) \to ((a = \{y\}) \vee (a = \{y, z\}))) \wedge (\forall a''((a'' = \{y, z\}) \to a'' \in x))$

$\quad\, \| $

$\{\{y\}, \{y, z\}\}$

a shorter one $\forall a\left((a\in x)\leftrightarrow\left((a=\{y\})\vee (a=\{y,z\})\right)\right)$

Can use same idea to make others shorter

eg

$x=\{y\}$ abbreviates $\forall z\left((z\in x)\leftrightarrow(z=y)\right)$

$x=\{y,z\}$ abbreviates $\forall a\left((a\in x)\leftrightarrow\left((a=y)\vee(a=z)\right)\right)$

We could have used a more consise language if we had wanted to

eg instead of

$A \leftrightarrow B$ we could write $(A \to B) \wedge (B \to A)$

eg instead of

$A \to B$ we could write $(\neg A) \vee B$

Can we be more consise?

In fact just need Nand

In logic it is often called the Sheffer stroke

$A \uparrow B = \neg(A \wedge B)$

③ Propositional calculus

Definition A propositional function on the letters $A_1, \dots, A_n$ is a string of symbols defined as follows:

① Each A_i is a propositional function

② If P and Q are propositional functions then so are $(P \wedge Q)$, $(P \vee Q)$, $(\sim P)$, $(P \to Q)$ and $(P \leftrightarrow Q)$

Idea this captures the part of our formal language with no quantifiers $(\forall, \exists)$ and where each A_i will ultimately be a well formed formula, moreover a sentence

Note these A_i are differant from our variables in our formal language $\longrightarrow$ in particular they will ultimately be replaced with well formed formulas

We think of these as functions taking n tuples of truth values

eg $n=3$ (T, T, F)

and returning a truth value

eg $A_1 \wedge (A_2 \vee A_3)$

sub in

get $T \wedge (T \vee F)$

$= T \wedge T = T$

As functions they are defined by the following truth tables

A_1	$\neg A_1$
F	T
T	F

$A_1 \downarrow$	output
F	T
T	F

input

$\neg A_1$

$A_1 \downarrow$ \ $A_2 \rightarrow$	F	T
F	F	F
T	F	T

$A_1 \wedge A_2$

$A_1 \downarrow$ \ $A_2 \rightarrow$	F	T
F	F	T
T	T	T

$A_1 \vee A_2$

$A_1 \downarrow$ \ $A_2 \rightarrow$	F	T
F	T	T
T	F	T

$A_1 \rightarrow A_2$

$A_1 \downarrow$ \ $A_2 \rightarrow$	F	T
F	T	F
T	F	T

$A_1 \leftrightarrow A_2$

A_1	A_2	$A_1 \rightarrow A_2$	$\neg A_1$	$(\neg A_1) \vee A_2$
F	F	T	T	T
F	T	T	T	T
T	F	F	F	F
T	T	T	F	T

same

as an eg lets check

$A_1 \rightarrow A_2$ is the same as $(\neg A_1) \vee A_2$

eg Use a truth table to evaluate

$$(\neg(A \wedge B)) \vee (A \vee B)$$

A	B	$A \vee B$	$A \wedge B$	$\neg(A \wedge B)$	the whole thing
F	F	F	F	T	T
F	T	T	F	T	T
T	F	T	F	T	T
T	T	T	T	F	T

a propositional function which is always true is called identically true

eg Show $A \vee (\neg A)$ is always true using a truth table

A	$\neg A$	$A \vee (\neg A)$
F	T	T
T	F	T

If a propositional function is always true we say it is identically true.

We want rules to deduce valid statements. These should match our intuitive notion of true

Rule A Let P be a propositional function in the letters $A_1, A_2, \dots, A_n$. If P is identically true then P with each A_i replaced by any sentence is a valid statement

Rule B If A and $A \to B$ are valid statements then so is B

These together give us propositional calculus. There is another way to understand it

→ more rules

→ no truth tables

Reductio ad absurdum (negation introduction)

From p and [accepting q leads to a proof that $\neg p$], infer $\neg q$.

Double negative elimination

From $\neg\neg p$, infer p.

Conjunction introduction

From p and q, infer $(p \wedge q)$.

From p and q, infer $(q \wedge p)$.

Conjunction elimination

From $(p \wedge q)$, infer p.

From $(p \wedge q)$, infer q.

Disjunction introduction

From p, infer $(p \vee q)$.

From p, infer $(q \vee p)$.

Disjunction elimination

From $(p \vee q)$ and $(p \rightarrow r)$ and $(q \rightarrow r)$, infer r.

Biconditional introduction

From $(p \rightarrow q)$ and $(q \rightarrow p)$, infer $(p \leftrightarrow q)$.

Biconditional elimination

From $(p \leftrightarrow q)$, infer $(p \rightarrow q)$.

From $(p \leftrightarrow q)$, infer $(q \rightarrow p)$.

Modus ponens (conditional elimination)

From p and $(p \rightarrow q)$, infer q.

Conditional proof (conditional introduction)

From [accepting p allows a proof of q], infer $(p \rightarrow q)$.

(from Wikipedia "propositional calculus")

These are the usual deduction rules for propositional calculus.

We get then as, for example,

$$((p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)) \rightarrow r$$

if you put this in a truth table you would get that is is identically true

so by rule (A) this is valid

so if we knew $p \vee q$, and $p \rightarrow r$, and $q \rightarrow r$

then use conjunction introd.

to get $(p \vee q) \wedge (p \to r) \wedge (q \to r)$

and then rule B

get r is valid

But what about conjunction introd.?

Same idea.

You can check that everything above comes from an identically true implication

Tougher is showing the 11 rules above are enough to derive all identically true propositional functions.

4 Next time

The liar's paradox