# Math 303, Fall 2011, Lecture 22

## ① Cantor diagonalization

It would be easy to wonder from last time if anything is larger than countable.

But there is

$\boxed{\text{Cantor's theorem} \quad \text{Let } X \text{ be any set, then } X \prec \mathcal{P}(X)}$

proof The function $f : X \to \mathcal{P}(X)$
$f(x) = \{x\}$ is one-to-one
so $X$ is equivalent to a subset of $\mathcal{P}(X)$.
thus $X \preceq \mathcal{P}(X)$

It remains to show that $X \not\sim \mathcal{P}(X)$

Suppose to the contrary $g : X \to \mathcal{P}(X)$
is one-to-one and onto

let $A = \{ x \in X : x \notin g(x) \}$

then $A \subseteq X$ so $A \in \mathcal{P}(X)$

but $g$ is onto so there is an $a \in X$

so that $g(a) = A$

is $a \in A$?

if $a \in A$ so $a \notin g(a) = A$ contradiction

if $a \notin A$ but $A = g(a)$ so $a \in A$ contradiction

in both cases we get a contradiction

So there is no one-to-one and onto $g : X \to \mathcal{P}(X)$

thus $X \not\approx \mathcal{P}(X)$ so $X \prec \mathcal{P}(X)$

Notes and consequences

1. To get our contradiction we embedded something rather like Russell's paradox into this set up

(2) $\mathcal{P}(X) \sim 2^X$ ⟵ exponentiation of sets in the sense of the set of functions from $X$ to $2$ not ordinal exponentiation.

by

$$f: \mathcal{P}(X) \to 2^X$$

$f(Y) =$ the function that takes $x \in X$ to $\begin{cases} 0 & \text{if } x \notin Y \\ 1 & \text{if } x \in Y \end{cases}$

$Y \subseteq X$

$f$ is one-to-one and onto (check)

so another way to phase Cantor's theorem is $\boxed{X \prec 2^X}$

$(X \prec \mathcal{P}(X) \sim 2^X$ so $X \prec 2^X)$

(3) This result is often called Cantor Diagonalization

what is diagonal about it?

say $X$ is countable. List its elements

$x_0$

$x_1$

$x_2$

$x_3$

$x_4$

$\vdots$

make a table for $g(x)$

$g: X \to \mathcal{P}(X)$ one-to-one and onto

(goal show a contradiction hence there is no such $g$)

| $x$ | $g(x)$ |
|---|---|
| $x_0$ | $g(x_0) \subseteq X$ |
| $x_1$ | $g(x_1) \subseteq X$ |
| $x_2$ | $g(x_2) \subseteq X$ |
| $\vdots$ | |

rephrase this using the bitstring representation of $g$ (ie using $\mathcal{P}(X) \sim 2^X$)

say ..

| $x$ | $g(x)$ |
|---|---|
| $x_0$ | $0, 0, 1, 1, 0, \ldots$ |
| $x_1$ | $1, 0, 0, 0, \ldots$ |
| $x_2$ | $1, 1, 1, 0, \ldots$ |
| $\vdots$ | |

means $x_0 \notin g(x_0)$

means $x_1 \notin g(x_0)$

means $x_3 \in g(x_0)$

means $x_2 \in g(x_0)$

means $x_0 \in g(x_1)$

means $x_2 \in g(x_2)$

diagonal hence diagonalization argument.

What is $A$? we had $A = \{x \in X : x \notin g(x)\}$

$1, 1, 0$ ← row for $A$

so the row for $A$ differs from the $i^{th}$ row of the table at the $i^{th}$ position

so $A$ can't appear in the table. So $g$ can't be onto, that's our contradiction in this context.

4 One special case is $\mathbb{Q} \prec \mathbb{R}$

set of rational numbers (for $\mathbb{Q}$) set of real numbers (for $\mathbb{R}$)

first note $\mathbb{Q} \sim \omega$

2 ways to do it

1 $0,\ 1,\ -1,\ \frac{1}{2},\ -\frac{1}{2},\ \frac{3}{2},\ -\frac{3}{2},$

2

$$\begin{array}{cccc}
0, 1, -1 & 2, -2 & 3, -3 & 4, -4 \\
\frac{1}{2}, -\frac{1}{2} & & \frac{3}{2}, -\frac{3}{2} & \\
\frac{1}{3}, -\frac{1}{3} & \frac{2}{3}, -\frac{2}{3} & & \frac{4}{3}, -\frac{4}{3}
\end{array}$$

What about $\mathbb{R}$? $2^{\omega} \preceq \mathbb{R}$

view any element of $\mathbb{R}$ as a decimal expansion view any element of $2^\omega$ as a string of 0s and 1s. Just put a decimal dot in front of the string to get a map from $2^\omega$ to a subset of $\mathbb{R}$

(tricky point - why didn't I use binary expansions?)

In fact $$\mathbb{R} \preceq 2^\omega$$

this time use binary expansions

but $0.1111111111111\ldots = 1$ as binary expansions

(same as fact that $0.99999\ldots\ldots = 1$ in decimal expansions)

so choosing binary expansions which don't end with an infinite string of 1s get

$$\mathbb{R} \preceq 2^{\mathbb{Z}} \sim 2^\omega$$

so $$\mathbb{R} \sim 2^\omega$$

so $$\mathbb{Q} \sim \omega \prec 2^\omega \sim \mathbb{R} \quad \text{so} \quad \mathbb{Q} \prec \mathbb{R}$$

② Next time

- Cardinals
- More Paradoxes and other crazy facts.

Note why is $0.9999\ldots = 1$ ?

what does it mean for $0.d_1 d_2 d_3 \ldots = x$

where $x \in \mathbb{R}$, $d_i$ are decimal digits

this means $|x - 0.d_1 d_2 \ldots d_n| \xrightarrow[n\to\infty]{} 0$

equiv $\lim_{n\to\infty} 0.d_1 d_2 \ldots d_n = x$

but apply this to $0.99999\ldots$

$$\Big|1 - 0.\underbrace{99\ldots9}_{n\ 9s}\Big| = 0.\underbrace{00000\ldots0}_{n-1\ 0s}1$$

take limit as $n \to \infty$ $\quad 0.\underbrace{0\ldots0}_{n-1\ 0s}1 \to 0$

so $0.99\ldots = 1$