Math 303, Fall 2011, Lecture 23

① Cardinal numbers

We want

We have

Definition A **cardinal number** is an ordinal number X such that if Y is an ordinal number and $X \sim Y$, then $X \leq Y$

If Z is any set, then $\#Z$, the cardinality of Z is the least ordinal equivalent to Z

this is

eg What is $\#3$? (it had better be 3 or this theory is really stupid)

The same holds for every natural number

eg What is $\#\mathbb{Z}$? set of integers

When we view ω as a cardinal we usually give it a different name

this is $\aleph_0$

Some facts about cardinals

① Let X and Y be sets

$$\#X = \#Y \iff X \sim Y$$

proof Any set is equivalent to its cardinal number

so

thus if $\#X = \#Y$

while if $X \sim Y$

2 There is an ordering on cardinals which makes any set of cardinals well ordered

proof

② The continuum hypothesis

How do we count with cardinals?

Let $\aleph_1$ be

We also know

Certainly

The continuum hypothesis says $\aleph_1 = 2^{\aleph_0}$

If not then

Why the word continuum?

Georg Cantor formulated the continuum hypothesis in 1874

In 1900 determining whether the continuum hypothesis was true or false was the first of Hilbert's problems

The resolution though was far from what Hilbert would have imagined

In 1940 Kurt Gödel showed that the continuum hypothesis is consistent with the axioms of Zermelo-Fraenkel set theory

In 1963 Paul Cohen showed that the negation of the continuum hypothesis is also consistent with the axioms of Zermelo-Fraenkel set theory.

Thus

THE CONTINUUM HYPOTHESIS IS

INDEPENDENT

OF THE AXIOMS OF SET THEORY

I think this is one of the craziest results of modern mathematics

Did you notice that the Halmos book is too old to know this?

It is about a semester of work to go through the proof and another to build some model theoretic background.

(One source of the proof is the last third or so of Cohen's book)

lets have a break now

③ Skolem's paradox

from lecture 18 we had

⑥ (Löwenheim - Skolem) Let T be a set of constant symbols and relation symbols.
Let M be a model of these symbols
Then there is an elementary submodel N of M with

$$\#(N) \le \begin{cases} \#(T) & \text{if } \#(T) \ge \#(\omega) \\ \#(\omega) & \text{if } \#(T) < \#(\omega) \end{cases}$$

(ie $S = \emptyset$ but have an interpretation of these symbols)

where an elementary submodel is one in which the same sentences are true

Now lets apply this to set theory

In set theory we have

So by Löwenheim - Skolem

But from Cantor diagonalization

Thus

this is Skolem's Paradox

How do we resolve Skolens paradox?

Skolem pointed out this paradox in 1922

Skolem felt that this result showed that first order logic was not the right tool for the foundations of mathematics.

So far history has disagreed, but the picture is certainly not as clean as people hoped at the time

But the lack of cleanness – the paradoxes – are beautiful in their own right hence this course.

And that's the end. Thanks for coming. Monday will be review for the final.