Math 303, Fall 2011, Lecture 3

① Power sets

We've been considering the subsets of a set E. Do all such subsets form a set?

Axiom of Power sets
For every set E, there is a set $\mathcal{P}$ consisting of precisely the subsets of E

that is $A \subseteq E$ if and only if $A \in \mathcal{P}$

We write $\mathcal{P}(E)$ for the power set of E

eg what is $\mathcal{P}(\{1,2,3\})$?

$$= \{ \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \phi, \{1,2,3\} \}$$

eg what is $\mathcal{P}(\{a\})$?

$$= \{\phi, \{a\}\}$$

eg what is $\mathcal{P}(\phi)$?

$$= \{\phi\}$$

Question. If E has n elements how many elements does $\mathcal{P}(E)$ have?

in examples $n=3$ $\mathcal{P}(E)$ had 8 elemt

$n=1$ $\mathcal{P}(E)$ had 2 elmt

$n=0$ $\mathcal{P}(E)$ had 1 elemt.

maybe 2^n

$_nC_k$

notice there are $\binom{n}{k}$ k-element subsets

and $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ binomial id.

another way assume ok for $n-1$, let E be such a set

then add one more element.

$E \cup \{a\}$, $a \notin E$

then $\mathcal{P}(E \cup \{a\}) = \mathcal{P}(E) \cup \mathcal{D}$,

where $\mathcal{D} = \{ A \cup \{a\} \mid A \in \mathcal{P}(E) \}$

but $|\mathcal{P}(E)| = 2^{n-1}$ $\quad |\mathcal{D}| = 2^{n-1}$

no overlaps so $|\mathcal{P}(E \cup \{a\})| = 2^n$

another way the i^{th} element of E is

in a given subset or not. 2 possibilities

independent so # subsets is $\underbrace{2 \cdot 2 \cdots 2}_{n} = 2^n$

(2) Constructing things part 1 — intersections and complements.

These sets and axioms are ok, but if we need a new axiom for each new construction that isn't very good.

Fortunately we can already do stuff with what we have

Define $$\boxed{A \cap B = \{ a \mid a \in A \text{ and } a \in B \}}$$

eg $\{1, 2, 3, 4\} \cap \{2, 4, 6\} = \{2, 4\}$

eg $\{\{\phi\}, \phi\} \cap \phi = \phi$

but wait that definition is no good at all!

to avoid Russell's paradox we were only allowed to specify subsets

all definitions must be of the form

$$\{x \in A : x \text{ satisfies some property}\}$$

Fix $\boxed{A \cap B = \{x \in A \mid x \in B\}}$

but this doesn't look symmetric in A and B even though we know it is from the other def.

Properties

(1) What is $A \cap \emptyset$? $\emptyset$

(2) What is $A \cap A$? A

(3) What is $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

that's a little harder

Suppose $x \in A \cap (B \cup C)$

that means $x \in A$ and $x \in B \cup C$

that is $x \in A$ and $(x \in B$ or $x \in C)$

so $(x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$

so $(x \in A \cap B)$ or $x \in (A \cap C)$

so $x \in (A \cap B) \cup (A \cap C)$

This shows $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

Likewise if $x \in (A \cup B) \cup (A \cap C)$

then

do it yourself

So $x \in A \cap (B \cup C)$

Therefore $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

This should remind you of $a(b+c) = ab + ac$

for addition and mult

But for sets it is also true that

(4) $$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

(but not $a + (bc) \overset{?}{=} (a+b)(a+c)$!)

NO

can you prove it in the same way?

More properties

(5) $$A \cap B = B \cap A$$

(6) $$A \cap (B \cap C) = (A \cap B) \cap C$$

What about extending intersections to sets of sets the way we did for unions

Define For any nonempty set $\mathcal{C}$

$$\cap \mathcal{C} = \bigcap_{B \in \mathcal{C}} B = \{ x \in A \mid x \in B \text{ for every } B \in \mathcal{C} \text{ where } A \text{ is any element of } \mathcal{C} \}$$

eg $\mathcal{C} = \{ \{a, b\}, \{b, c\}, \{a, b, c\} \}$

$\cap \mathcal{C} = \{ b \}$

$\{\}$

Question for the break

What is $\cap \emptyset$?

$\bigcap \emptyset$ should be ??

what is not in $\bigcap \emptyset$?

if $x \notin \bigcap \emptyset$ then there is at least one element of $\emptyset$ which does not contain x

but $\emptyset$ has no elements so no way to have $x \notin \bigcap \emptyset$

So $\bigcap \emptyset$ would have to be everything

But Russell's paradox said we don't have a set of everything.

So $\bigcap \emptyset$ is just not allowed – it is undefined.

The complement

Let A and B be sets then the **set difference** or **relative complement** of B in A is

$$A - B = \{x \in A : x \notin B\}$$

eg $A = \{1, 2, 98\}$ $\quad B = \{2\}$

$$A - B = \{1, 98\}$$

eg $A = \{1, 2, 98\}$ $\quad B = \{2, 4\}$

$$A - B = \{1, 98\}$$ ← definition did not require $B \subseteq A$

eg $A - A = \emptyset$

Sometimes we will be taking a lot of complements in the same outer set. Then the following notation is more convenient

$\boxed{\text{Fix a set } E. \quad \text{Let } A' = E \setminus A, \text{ the complement of } A \text{ (in } E)}$

eg $E = \{a, b, c, d\}$, $A = \{b, c\}$

then $A' = \{a, d\}$

eg $E = \mathbb{Z}_{>0}$ the set of positive integers

A = set of even (positive) integers

A' = the set of odd positive integers

Try these. Fix a set E

	suppose $A \subseteq E$	if not.
① What is $(A')'$	A	$A \cap E$
② What is ϕ'	E	E
③ What is E'	ϕ	ϕ
④ What is $A \cap A'$	ϕ	ϕ
⑤ What is $A \cup A'$	E	$A \cup E$

Here are some more

(6) $\boxed{A, B \subseteq \mathcal{E}}$ $A \subseteq B$ if and only if $B' \subseteq A'$

$\Rightarrow$ Suppose $A \subseteq B$. Then every $x \in A$ is also in B so if $y \notin B$ it is also the case that $y \notin A$

Take $y \in B'$ then $y \in \mathcal{E}$ $y \notin B$

Then $y \in \mathcal{E}$ $y \notin A$

so $y \in A'$

$\Leftarrow$ For the other direction use A', B' in place of B and A in the above argument (uses $(A')' = A$, $(B')' = B$)

(7) De Morgan's Laws

$$(A \cup B)' = A' \cap B'$$

$$(A \cap B)' = A' \cup B'$$

Suppose $x \in (A \cup B)'$ So $x \in \mathcal{E}$ and $x \notin A \cup B$

So $x \in \mathcal{E}$ and ($x \notin A$ and $x \notin B$)

so $x \in E$ and $x \notin A$ and $x \notin B$

so $x \in A'$ and $x \in B'$

so $x \in A' \cap B'$

Conclusion: $\boxed{(A \cup B)' \subseteq A' \cap B'}$

Now suppose $x \in A' \cap B'$.

so $x \in A'$ and $x \in B'$

so $(x \in E$ and $x \notin A)$ and $(x \in E$ and $x \notin B)$

so $x \in E$ and $\boxed{x \notin A \text{ and } x \notin B}$

so $\boxed{x \in E \text{ and } x \notin A \cup B}$

so $x \in (A \cup B)'$ Conclude $\boxed{A' \cap B' \subseteq (A \cup B)'}$

Similarly for the other one.

gives $(A \cup B)' = A' \cap B'$

③ Next time

ordered pairs and cartesian products

$\boxed{\text{please read Halmos section 6}}$