Math 303, Fall 2011, Lecture 7

① Functions

Suppose X and Y are sets. We would like to encode functions

$$\underbrace{f}_{\text{name of function}} : \underbrace{X}_{\text{domain}} \to \underbrace{Y}_{\text{codomain}}$$ using sets

How can we do this?

Use ordered pairs

Just let f be the set of (x, y) where $f(x) = y$

To make this a precise definition we need

$$f = \{ (x,y) \in ?? \mid ?? \}$$

and so $f(x) = y$ must be writable in logic

Which subsets S of $X \times Y$ are functions?

for each $x \in X$ we need a unique $y \in Y$ so $(x,y) \in S$

(Need at least one as otherwise $f(x)$ is not defined, and no more than one since $f(x)$ can only have one value)

$$\boxed{\text{Define} \quad Y^X = \{ f \in \mathcal{P}(X \times Y) \mid f \text{ is a function} \}}$$

eg let $X = Y = \omega \; (= \{0, 1, 2, \dots\})$

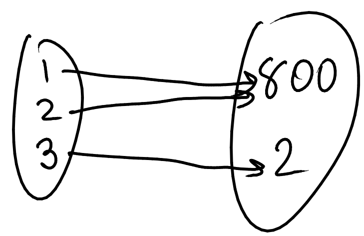
let $f(x) = x^2$

What is f as a set?

Answer $\{(0,0), (1,1), (2,4), (3,9), (4,16), \dots\}$

eg let $X = \{1, 2, 3\}$ $\quad Y = \{800, 2\}$

let f be



What is f as a set?

Answer $\{(1,800), (2,800), (3,2)\}$

eg let $X = \emptyset$, let $Y = \omega$

What can f be?

Answer $f = \emptyset$ the function does nothing since its domain is empty eg What is $Y^{\emptyset}$ for any set Y

Answer $\emptyset$

eg let $X = \{1, 2\}$ let $Y = \{3, 4\}$

what is Y^X?

Answer

$1 \to 3$ $2 \to 3$	$1 \to 4$ $2 \to 4$	$1 \to 3$ $2 \to 4$	$1 \to 4$ $2 \to 3$

$$Y^X = \{ \{(1,3),(2,3)\} \quad \{(1,4),(2,4)\} \quad \{(1,3),(2,4)\} \quad \{(1,4),(2,3)\} \}$$

lets remember some function words

domain $\quad \operatorname{dom} f = \{x : \text{for some } y \ (x,y) \in f\}$

range $\quad \operatorname{ran} f = \{y : \text{for some } x \ (x,y) \in f\}$

If $\operatorname{ran} f = Y$ then f is onto Y

If $X \subseteq Y$ then the function $f: X \to Y$ defined by $f(x) = x$ for all x in X is called the inclusion map

The inclusion map from X into X is called the identity map

The function $f: X \times Y \to Y$ given by $f(x,y) = y$ is the projection map onto the second coordinate we also have

$f: X \times Y \to X$ given by $f(x,y) = x$

the projection map onto the first coordinate

If f maps distinct elements onto distinct elements

then f is one-to-one

written in function language f one-to-one means

if $f(x_1) = f(x_2)$ then $x_1 = x_2$

written in set language f one-to-one means

if $(x_1, y) \in f$ and $(x_2, y) \in f$

then $x_1 = x_2$

eg Let $X = \{1, 2\}$ $Y = \{1, 3\}$

what is projection from $X \times Y$ to Y?

Answer $f(x, y) = y$ for all $(x, y) \in X \times Y$

As a set

$$\{\{(1,1), 1\}, \{(1,3), 3\}, \{(2,1), 1\}, \{(2,3), 3\}\}$$

eg is the projection in the previous example onto Y?

Answer, yes

is it one-to-one?

Answer, no

② Finite and infinite (This is back to Halmos p52)

Definition Say two sets A and B are equivalent or the same size if there is a one-to-one and onto function from A to B

could mean other things in other books so watch out.

this is what a one-to-one correspondence is

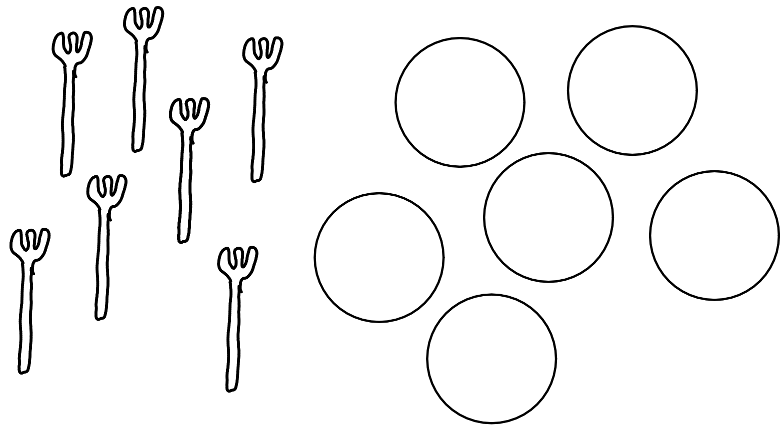
eg is $\{2, 3, 8, 12\}$ equivalent to 4?

yes $4 = \{0, 1, 2, 3\}$

let $f = \{(0,2), (1,3), (2,8), (3,12)\}$

then f is one-to-one and onto

The idea here is that if I want to know if two sets have the same number of elements then I don't have to know how to count the elements, I just have to match them up and see if I get any leftovers eg



For the break

find an example of a set which is equivalent to a proper subset of itself eg $\quad f : \omega \to \omega$, $f(n) = n^+$ $\quad$ or $\quad f : \omega \to \omega$, $f(n) = n^2$ $\quad$ etc.

Fortunately this doesn't happen for individual natural numbers

Claim: If $n \in \omega$ then n is not equivalent to a proper subset of n

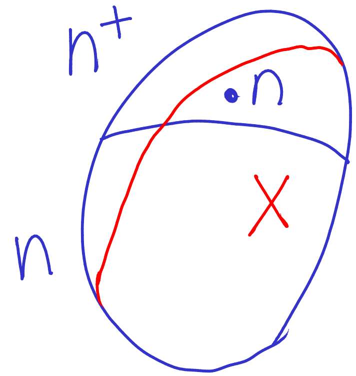
proof By induction: Let S be the set of $n \in \omega$ which are not equivalent to any proper subset of themselves.

$\emptyset \in S$ since $\emptyset$ has no proper subset

Take $n \in S$ and say $f : n^+ \to X$ is one-to-one and onto with $X \subsetneq n^+$

But $n^+ = n \cup \{n\}$

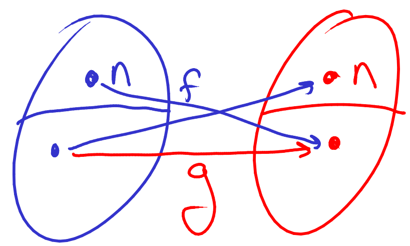
if $n \in X$



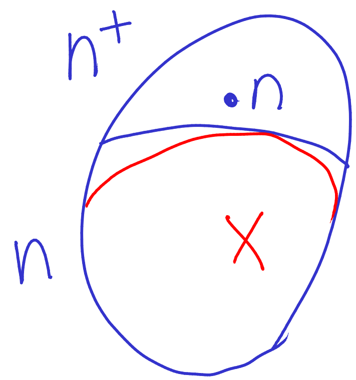
then $X \setminus \{n\} \subsetneq n$
so we just need to find a one-to-one, onto map $g : n \to X \setminus \{n\}$ to get a contradiction to $X \in S$

if $f(n) = n$ let g be f restricted to $n^+ \setminus \{n\} = n$

if $f(n) \neq n$ let $g(a) = f(a)$ for $f(a) \neq n$
and $g(a) = f(n)$ for $f(a) = n$



if $n \notin X$



then let g be f restricted to $n^+ \setminus \{n\} = n$

then g maps onto $X \setminus \{f(n)\}$

so $X \setminus \{f(n)\} \subsetneq n$

again contradicting $n \in S$

all together we get that if $n \in S$ then $n^+ \in S$
so by the principle of mathematical induction $S = \omega$
and so no natural number is equivalent to a proper
subset of itself.

Claim A set can be equivalent to at most one natural number

proof Suppose S is equivalent to n and to m
with $n \neq m$, $n, m \in \omega$.
Then n is equivalent to m (can you explain why?)
But either $n \subsetneq m$ or $m \subsetneq n$
and so this contradicts the previous claim.

Now we can define a set A to be finite if it is equivalent to some natural number, and infinite otherwise

Also define the size or number of elements of a finite set A to be the unique natural number equivalent to A

use the notation $\#A$ for the size of A

This notion of size corresponds to our usual notion of size

for example if $A \subseteq B$ then $\#A \leq \#B$

proof Say B is equivalent to n by the map f

Then f restricted to A gives A equivalent to a subset X of n. Suppose $\#X > n$

Then we would have a

one-to-one correspondence between $\#X$ and x, but $x \subseteq n \subsetneq \#X$ so we would have a one-to-one correspondence between $\#X$ and a proper subset of $\#X$, contradictio[n]

③ Next time

- Summary of our axioms so far and outlook
- The axiom of choice

$\boxed{\text{Please read Halmos Section 15}}$