Generating functions

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1 Enumeration

Perhaps the most fundamental question about a combinatorial class, is "how many?". The enumeration problem can be extremely useful for solving other problems, and our approach will use generating functions since they are so extremely powerful. We will only just scratch the surface of their potential.

Our initial approach will build up a small toolbox of combinatorial constructions

- union
- cartesian product
- sequence
- set and multiset
- cycle
- pointing and substitution

which translate to simple (and not so simple) operations on the generating functions.

1.1 Generating functions

We now jump head first into generating functions. Wilf's book *generatingfunctionology* is a great first text if you want more details and examples.

Definition. The ordinary generating function (OGF) of a sequence (A_n) is the formal power series

$$A(z) = \sum_{n=0}^{\infty} A_n z^n.$$

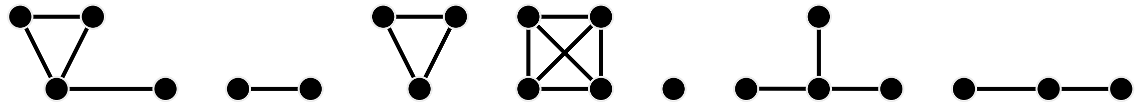
We extend this to say that the ogf of a class $\mathcal{A}$ is the generating function of its counting sequence $A_n = \text{cardinality}(\mathcal{A}_n)$. Equivalently, we can write the OGF as

$$A(z) = \sum_{\alpha \in \mathcal{A}} z^{|\alpha|}.$$

In this context we say that the variable z marks the size of the underlying objects.

Exercise. Prove that these two forms for $A(z)$ are equivalent.

For example, consider the class of graphs $\mathcal{H}$ given here



If we define size to be the number of vertices, then the generating function is simply

$$H(z) = z^4 + z^2 + z^3 + z^4 + z^1 + z^4 + z^3 = z + z^2 + 2z^3 + 3z^4.$$

On the other hand, if we define size to be the number of edges, then the ogf is

$$H(z) = z^4 + z^1 + z^3 + z^6 + z^0 + z^3 + z^2 = 1 + z + z^2 + 2z^3 + z^4 + z^6.$$

In both these cases we are treating the generating function as some sort of projection in which we replace each vertex (edge) by a single atomic "z" and forget all detailed information about the object and then just gather the monomials together to get the ogf.

Thus our examples become

$$\text{Binary words:} \quad W(z) = \sum_{n=0}^{\infty} 2^n z^n = \frac{1}{1-2z}$$

$$\text{Permutations:} \quad P(z) = \sum_{n=0}^{\infty} n! z^n$$

The first of these is a simple geometric sum. The second function does not have any simple expression in terms of standard functions. Indeed it does not converge (as an analytic object) for any z complex z other than 0. However, we can still manipulate it as a *purely formal* algebraic object.

We only care about convergence in that we require that that the coefficient of any finite power of z is finite. This brings us to even more notation:

Definition. Let $[z^n]f(z)$ denote the coefficient of z^n in the power series expansion of $f(z)$

$$[z^n]f(z) = [z^n]\left(\sum_{n=0}^{\infty} f_n z^n\right) = f_n$$

2 Manipulating formal power series

2.1 Definitions

Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers. We call the (possibly infinite) sum $a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \ldots$ a **formal power series**. Here x is a variable, but we do not apriori care if the sum makes sense for any value of x aside from 0. Despite this, we can consider it a function of x and write it as $A(x)$. We say that a_k is the *coefficient* of x^n in $A(x)$, and we make use of the shorthand

$$[x^k]A(x) = a_k.$$

The sum is said to be *formal* because we cannot collapse any of the terms. So, if $a_0 + a_1 x + a_2 x^2 = b_0 + b_1 x + b_2 x^2$, then it must be that $a_0 = b_0$, $a_1 = b_1$ and $a_2 = b_2$. There is a single power series equal to 1: $1 = 1 + 0x + 0x^2 + \ldots$, and a single one equal to 0: $0 + 0x + 0x^2 + \ldots$.

The formal power series turns out to be a very convenient way to store a sequence. We will see that we can store infinite sequences with a very small amount of memory using what we know about functions formal power series which arise as Taylor series expansions around 0.

2.2 Sum and Product

A formal power series is a mathematical object which behaves essentially like an infinite polynomial. We can define addition and multiplication of power series. Let $A(x) = \sum_{n\geq 0} a_n x^n$ and $B(x) = \sum_{n\geq 0} a = b_n x^n$. Then

$$A(x) + B(x) := \sum_{n\geq 0} (a_n + b_n)\, x^n$$

$$A(x) \cdot B(x) := \sum_{n\geq 0} \sum_{0\leq k\leq n} a_k b_{n-k} x^n.$$

Remark, we have completely specified the coefficient of x^n for each n, and so the sum and product are well defined. They can be computed in finite time given $A(x)$ and $B(x)$.

2.3 Multiplicative inverse $A(x)^{-1}$

The multiplicative inverse of a formal power series $A(x)$ is the formal power series $C(x) = \sum_{n\geq 0} c_n x^n$ that satisfies

$$A(x) \cdot C(x) = 1.$$

Thus,

$$\sum_{n\geq 0} \sum_{k\geq n} a_k c_{n-k} x^n = 1 + 0x + 0x^2 + \dots.$$

Recall that two formal power series are equal if and only all of their coefficients are the same. This leads to the system of equations:

$$\begin{aligned} a_0 c_0 &= 1 && (1)\\ a_1 c_0 + a_0 c_1 &= 0 && (2)\\ a_2 c_0 + a_1 c_1 + a_0 c_2 &= 0 && (3)\\ &\vdots && (4) \end{aligned}$$

This is a triangular system, and hence we can compute c_k once we know of $c_0, \dots, c_{k-1}$. We see from (1) that $c_0 = 1/a_0$, and provided that $a_0 \neq 0$, this is well-defined.

Example: The inverse of $A(x) = 1 - x$. The above system simplifies to:

$$\begin{aligned} c_0 &= 1\\ -c_0 + c_1 &= 0\\ -c_1 + c_2 &= 0\\ &\vdots\\ -c_k + c_{k+1} &= 0\\ &\vdots \end{aligned}$$

This has the unique solution: $c_n = 1$ for all n. We should recognize this as the *geometric series*:

$$(1 - x)^{-1} = \sum_{n\geq 0} x^n = 1 + x + x^2 + x^3 + \dots.$$

2.4 Composition

We define composition of formal power series as follows:

$$A(B(x)) := \sum_{n\geq 0} a_n \cdot B(x)^n.$$

2.5 Differentiation

We can develop a complete theory of derivatives and integrals of formal power series using the power rule $\frac{d}{dx}x^k = kx^{k-1}$, and essentially generalizing the derivative of a polynomial. Remark, this does not involve a limit and so the usual properties (sum rule, product rule, chain rule) should be proved from the definition of formal power series.

- Definition: $\frac{d}{dx}A(x) = \sum_{n\geq 0}(n+1)a_{n+1}x^n$;
- Product rule: $\frac{d}{dx}; (A(x)B(x)) = \left(\frac{d}{dx}A(x)\right)B(x) + A(x)\left(\frac{d}{dx}B(x)\right)$;
- Chain rule for powers: $\frac{d}{dx}(A(x))^n = n(A(x))^{n-1}\frac{d}{dx}A(x)$.

Exercise. Prove the above product and chain rules.

2.6 Two special series: exp, log

The formal power series $\sum_{n\geq 0} x^n/n!$ is reminiscent of the Taylor series for the exponential, and hence we define

$$\exp(x) := \sum_{n\geq 0} x^n/n!.$$

In fact, it satisfies all of the usual properties of exp, but again, since it is *defined* as the sum, these properties should be proved from the basic formal power series properties:

- $\frac{d}{dx}\exp(x) = \exp(x)$;
- $\exp(F(x) + G(x)) = \exp(F(x)) \cdot \exp(G(x))$;
- $\exp(F(x))^{-1} = \exp(-F(x))$.

Similarly, we can define a series

$$\log(1-x)^{-1} := \sum_{n\geq 1}\frac{x^n}{n}.$$

Exercise. Prove the following properties which we can derive from formal power series definitions:

1. $\frac{d}{dx}\log(1-x)^{-1} = (1-x)^{-1}$;
2. $\log(\exp(A(x))) = A(x)$;
3. $\exp(\log(A(x))) = A(x)$, if a_0=1;
4. $\log(A(x) + B(x)) = \log(A(x)) + \log(B(x))$, if $a_0 = b_0 = 1$.

3 The basic toolbox for coefficient extraction

There are several key theorems to aid with coefficient extraction. If $A(z) = \sum_{n=0}^{\infty} a_n z^n$, then $[z^n]A(z) := a_n$.

Power rule

$$[z^n]A(pz) = p^n[z^n]A(z)$$

Reduction rule

$$[z^n]z^m A(z) = [z^{n-m}]A(z)$$

Sum rule

$$[z^n]\left(A(z) + B(z)\right) = [z^n]A(z) + [z^n]B(z)$$

Product rule

$$[z^n]A(z) \times B(z) = \sum_{k=0}^{n} \left([z^k]A(z)\right) \times \left([z^{n-k}]B(z)\right)$$

Binomial theorem Let n and k be positive integers.

$$[z^n](1+z)^r = \binom{r}{n} = \frac{r!}{(r-n)!n!}.$$

Extended binomial theorem Let k be a positive integer, and r be a real number. Then define

$$\binom{r}{k} := \frac{r(r-1)(r-2)\dots(r-k+1)}{k!}.$$

Then

$$[z^k](1+z)^r = \binom{r}{k}.$$

A Useful Binomial Formula

$$\begin{aligned}\binom{-r}{k} &= \frac{(-r)(-r-1)\dots(-r-k+1)}{k!}\\ &= (-1)^k\binom{r+k-1}{r-1} = (-1)^k\binom{r+k-1}{k}.\end{aligned}$$

Example (Catalan numbers)**.**
The generating function $\frac{1-\sqrt{1-4z}}{2z}$ is a "famous" generating function. The coefficients are called Catalan numbers, and we will revisit them shortly as a fundamental counting sequence.

$$\begin{aligned}[z^n]\frac{1-\sqrt{1-4z}}{2z} &= -\frac{1}{2}(-4)^n[z^n](1+z)^{1/2}\\ &= -\frac{1}{2}(-4)^n\binom{1/2}{n}\\ &= \frac{-(-4)^n}{2}\,\frac{1/2(1/2-1)(1/2-2)\dots(1/2-n+1)}{n!}\\ &= \frac{-(-4)^n}{2}(1/2)^n(-1)^{n-1}\frac{1\cdot1\cdot3\cdot5\dots(2n-3)}{n!}\\ &= 2^{n-1}\frac{1\cdot1\cdot3\cdot5\dots(2n-3)}{n!}\cdot\underbrace{\frac{(n-1)!}{(n-1)!}}_{=1}\\ &= \frac{1\cdot1\cdot3\cdot5\cdots(2n-3)\cdot\overbrace{(2\cdot n)\cdot(2\cdot(n-1))\cdot(2\cdot1)}^{2^{n-1}(n-1)!}}{n!(n-1)!}\\ &= \frac{(2n-2)!}{(n-1)!(n-1)!n}\\ &= \binom{2n-1}{n-1}\frac{1}{n}\end{aligned}$$

We will encounter this generating function more than once in this class. This generating function, as we shall see, is ubiquitous in combinatorics.

Exercise. Go to the web page: http://oeis.org/ This is the Online Encyclopedia of Integer sequences. The first few terms of this sequence are 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862. Enter them into the search engine and find some combinatorial interpretations of these numbers.

4 A sampling of counting sequences and generating functions

$\mathcal{A}$	A_n	$A(z)$
Permutations	$n!$	$\sum n! z^n$
All simple, labelled graphs	$2^{\binom{n}{2}}$	$\sum 2^{\binom{n}{2}} z^n$
Binary words	2^n	$\frac{1}{1-2z}$
Balanced parentheses	$\binom{2n}{n}\frac{1}{n+1}$	$\frac{1-\sqrt{1-4x}}{2}$

4.1 Advanced Ideas

- **Asymptotic Analysis.** The values for which $A(x)$ is not defined can give us much information about the *asymptotic growth* of the coefficient. Consider $A(x) = \frac{1}{1-2x} = \sum 2^n x^n$. The coefficient a_n grows like 2^n. Remark that $A(x)$ is not defined at $x = 1/2$. In fact, very often we can formalize a connection between the smallest *singularity* of the formal power series ρ (roughly, the smallest value for which $A(x)$ is not defined) and the growth of a_n: $a_n \approx 1/\rho^n$. Again, complex analysis is fundamental in this remarkable theory.