MATH 343, SPRING 2012, ASSIGNMENT 1

DUE THURSDAY JANUARY 17, 2013 IN CLASS

Do **any three** of the following four problems. If you do more than three, only the first three will be graded.

(1) Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ be a formal power series. Define

$$\frac{d}{dx}A(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

(a) Prove (using only properties of formal power series) that the product rule of calculus holds, that is if $A(x)$ and $B(x)$ are formal power series, prove that

$$\frac{d}{dx}(A(x)B(x)) = \left(\frac{d}{dx}A(x)\right) B(x) + A(x)\left(\frac{d}{dx}B(x)\right)$$

(b) Define the formal power series $\exp(x) = \sum_{n=0}^{\infty} x^n/n!$. Prove (using only properties of formal power series) that

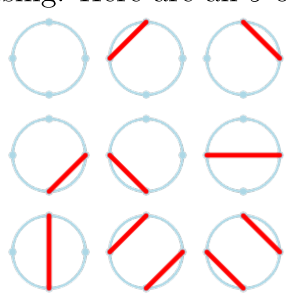
$$\frac{d}{dx}\exp(x) = \exp(x)$$

(2) (a) Give two different combinatorial classes with an infinite number of elements and the same underlying set.

(b) Give a third size function on the same set so that the result is not a combinatorial class

(3) We have several tools at our disposal to help understand and play with counting sequences. This exercise is to help you gain familiarity with some of these tools.

Let $\mathcal{M}$ be the following class of combinatorial objects: An object of size n is a circle on n points, with some collection chords between the vertices under the condition that no two chords are crossing. Here are all 9 objects of size 4:



(a) Prove that $\mathcal{M}$ satisfies the definition of combinatorial class.

(b) Determine by hand the first five elements of the counting sequence $(M_n)_{n\geq 0}$;

(c) The generating function of this sequence is:

$$M(z) = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z^2}.$$

Using some sort of computer algebra program, determine the first 10 elements of the counting sequence. Some examples: You can use the command `series` or `taylor` in Maple, or the command series in the prompt at the website `http://www.wolframalpha.com/`.

(d) Look up this sequence on the On-line encyclopedia of integer sequences (`http://oeis.org`). What is its sequence number? List two other combinatorial classes also counted by these numbers. Choose one, and draw all the elements of size less than or equal to 5.

(4) Let $\mathcal{B}$ be the class of binary rooted trees where every node has either 0 or 2 children.

(a) Find an equation which $B(x)$ satisfies.

(b) Determine $[x^n]B(x)$

(c) How does this compare to the class of all binary rooted trees from lecture?