Math 343 Lecture 4

① Basic combinatorial constructions

The value of generating functions is that they mirror the combinatorial structure of what you're counting in a useful way

Recall the binary rooted tree example

We went from

to

How do we make this more systematic?

This is today's goal

First $\boxed{\text{Union} \longleftrightarrow \text{Sum}}$

Prop let $\mathcal{B}$ and $\mathcal{C}$ be combinatorial classes with
then $\mathcal{A} = \mathcal{B} \cup \mathcal{C}$ is a combinatorial class
where for $a \in \mathcal{A}$,

proof

Note

$\mathcal{B} \cup \mathcal{C}$

prop Let $\mathcal{B}, \mathcal{C}$ be combinatorial classes with $\mathcal{B} \cap \mathcal{C} = \emptyset$
and let $\mathcal{A} = \mathcal{B} \cup \mathcal{C}$

Then $A(x) =$

proof

We've already seen a (slightly silly) example in the binary rooted trees:

Second $\boxed{\text{Cartesian product} \leftrightarrow \text{series product}}$

Recall **Def** let A and B be sets
then the cartesian product of A and B is

$$A \times B =$$

prop let $\mathcal{B}$ and $\mathcal{C}$ be combinatorial classes. Then
$\mathcal{A} = \mathcal{B} \times \mathcal{C}$ is a combinatorial class
where

proof

Note $\mathcal{B} \times \mathcal{C}$

Prop Let $\mathcal{B}$ and $\mathcal{C}$ be combinatorial classes and let $\mathcal{A} = \mathcal{B} \times \mathcal{C}$ then $A(x) =$

proof

We've also seen an example of this in the binary rooted trees

In the tree example we also needed the combinatorial class $\{\bullet\}$ and $\{\varepsilon\}$. To save writing curly brackets define the following

Def Let $\mathcal{E}$ be

ε

Let $\mathcal{Z}$ be

It can be useful to have

eg Returning again to the binary rooted trees

② Admissibility

Def Let Φ be a construction which takes as input m combinatorial classes $\mathcal{B}^{(1)}, \mathcal{B}^{(2)}, \dots, \mathcal{B}^{(m)}$ and builds as output a combinatorial class

$$\mathcal{A} = \Phi(\mathcal{B}^{(1)}, \mathcal{B}^{(2)}, \dots, \mathcal{B}^{(m)})$$

Then we say Φ is admissible iff

Admissibility tells us

Note

What about union? Is union admissible?

We don't want to extend the definition to non-disjoint unions as if $\mathcal{B} \cap \mathcal{C} \neq \emptyset$

then $|\mathcal{B}_n \cup \mathcal{C}_n| =$

Instead

Def let $\mathcal{B}$ and $\mathcal{C}$ be combinatorial classes

let $\mathcal{B} + \mathcal{C}$ be

This definition makes sense as

Also

and so

Finally with all this notation we would rewrite our tree decomposition as the combinatorial specification

There are many more admissible combinatorial constructions we'll discuss one more, but you can read the extra notes on some others.

③ The sequence construction

Think about Kleene star in regular languages

$a^* = \{ \qquad\qquad \}$

$\{a,b,c\}^* = \{ \qquad\qquad \}$

It gives all finite sequences

Let $\mathcal{A}$ be a combinatorial class. Suppose we want to construct all finite sequences of elements of $\mathcal{A}$

So we want

$\uparrow \qquad \uparrow \qquad \uparrow$

So define

Def Let $\mathcal{A}$ be a combinatorial class
write $\mathcal{A}^k$ for $\qquad\qquad\qquad\qquad \mathcal{A}^0$

Def Let $\mathcal{A}$ be a combinatorial class with
then $\mathrm{Seq}(\mathcal{A}) =$

This definition is not yet complete. What if we have
$\varepsilon \in \mathcal{A}$, $|\varepsilon| = 0$?

then

We will also use the notation

$Seq_k(A) =$

$Seq_{\geq k}(A) =$

and generally

eg

In order to find the generating function for $Seq(A)$ we need to talk about formal inverses, though as usual they behave just as you'd expect.

Def The multiplicative inverse of a formal power series $A(x)$ is the formal power series $C(x)$ such that

For this to be a good definition we need to know that $C(x)$ is unique if it exists. It's also useful to know when $C(x)$ exists. We can address both those points with the following calculation.

Suppose $A(x)C(x) = 1$

then

and

If $a_0 = 0$ then

If $a_0 \neq 0$ then

Thus **proposition** A formal power series $A(x)$ has an inverse iff $a_0 \neq 0$

proposition Let $A(x)$ be a formal power series with $a_0 = 0$ then

$$\frac{1}{1-A(x)} =$$

does this infinite sum of formal power series make sense?

proof

prop let $\mathcal{B}$ be a combinatorial class with $\mathcal{B}_0 = \emptyset$

let $\mathcal{A} = \operatorname{Seq} \mathcal{B}$ then

$A(x) =$

proof

④ Examples

eg Binary words, $\mathcal{B}$

so a specification is $\mathcal{B} =$

so $B(x) =$

eg Binary words where every block of 1s is of even length

$\mathcal{B} =$

so $B(x) =$

eg Binary words where no more than two 1s appear together

$\mathcal{B} =$

$B(x) =$

can you think of another way to represent it which looks more like this eg rooted trees where each node has any number of children but the children are ordered left to right (ie a sequence of children)

$$\mathcal{T} =$$

$$T(x) =$$

⑤ Next time

another example and a bijection