MATH 343, SPRING 2012
MIDTERM NOTES AND REVIEW QUESTIONS

Midterm notes

The midterm will be Thursday February 21 in class. It will cover everything we have done so far, including all the ranking and unranking stuff.

There is some choice in which questions you complete; you will need to complete 8 questions. Some questions are more theoretical, some are more computational. Proofs from class are fair game, as are algorithms. The homework questions are generally good midterm review questions. The only exception is that you will not have a computer during the exam so you will not be asked to work with `combstruct` or compute large terms.

Midterm review questions

Here are some more midterm review questions.

(1) Let $\mathcal{W}$ be the class of binary strings with the property that there are exactly two 0s, and all nonempty blocks of 0s have length at least 3. Give a specification for $\mathcal{W}$ and a formula for its generating function.

(2) Define $\exp(x)$ to be the formal power series

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Let $A(x)$ and $B(x)$ be formal power series. Prove that

$$\exp(A(x) + B(x)) = \exp(A(x)) + \exp(B(x))$$

(3) Let $\mathcal{C}$ be the class of integer compositions with all parts odd and no repeated parts.
   (a) Give a specification for $\mathcal{C}$ and a formula for its generating function.
   (b) List all the elements of $\mathcal{C}$ of size 7.

(4) Let $\mathcal{B}$ be the class of binary strings where every block of 0s is followed by a block of 1s of the same length (an initial block of 1s of any length is possible).
   (a) Find a formula for $T(x)$.
   (b) Find $[x^n]T(x)$.

(5) Let $\mathcal{M}$ be the class of diagrams from assignment 1 question 3 (note if this question was on the midterm I would define the class – you don't need to memorize which question is which from each assignment). Let $\mathcal{N}$ be the class of paths beginning at $(0,0)$, ending at $(n,0)$, using steps $(1,1)$, $(1,-1)$, and $(1,0)$ and never going strictly below the x-axis.
   (a) Calculate the first three terms of the counting sequences of $\mathcal{M}$ and $\mathcal{N}$.
   (b) Show that the counting sequences of $\mathcal{M}$ and $\mathcal{N}$ are the same by finding matching specifications for the two classes.

(6) Let $\mathcal{T}$ be the class of rooted trees where every vertex has a number of children which is either even or divisible by 3.

(a) Give a specification for $\mathcal{T}$.
(b) Give a closed formula for $[x^n]T(x)$.

(7) Let $\mathcal{T}$ be the class of rooted trees with red, blue, and green vertices where each red vertex has an even number of blue children and no other children, each blue vertex has at most 2 children of any colour, and each green vertex has only red children.
(a) Give a specification for $\mathcal{T}$
(b) What is the dependency digraph of your specification?

(8) Let $F(x)$ and $G(x)$ be formal Laurent series. Prove that

$$[x^{-1}]F(x)\frac{d}{dx}G(x) = -[x^{-1}]G(x)\frac{d}{dx}F(x)$$

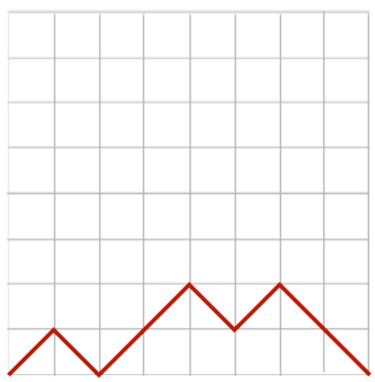
(9) Write pseudocode to describe an algorithm for Prüfer successor of labelled trees.

(10) Rank the subsets of size k of $\{1, 2, \ldots, n\}$ by listing the elements of each subset in increasing order and then ordering the subsets lexicographically. Let $L = (n_1, n_2, \ldots, n_k)$ be such a subset in increasing order. Prove that

$$\text{rank}(L) = \sum_{i=1}^{k} \sum_{j=n_{i-1}+1}^{n_i-1} \binom{n-j}{k-i}$$

where by convention $n_0 = 0$.

(11) Viewing a Dyck path as a word with each up step as 0 and each down step as 1 and ordering these words lexicographically, determine the rank of



(12) Rank the k-subsets of $\{1, 2, \ldots, n\}$ as described in question 10. Write pseudocode to describe an algorithm which given n and a k-subset in this form returns the rank of the subset.