MATH 817 ASSIGNMENT 1 SOLUTION

(1) Take any $a, b \in G$. $a^2 = b^2 = (ab)^2 = 1$ so $[a, b] = a^{-1}b^{-1}ab = (ab)(ab) = 1$.

(2) If $G = 1$ then G has exactly 1 subgroup. If there is any non-identity element of G which does not generate G then the subgroup generated by this element is a proper, nontrivial subgroup which is impossible. Thus $G = \langle x \rangle$ is cyclic. If G is infinite then $\langle x^2 \rangle$ is a proper nontrivial subgroup again a contradiction. Let $1 < n = |G|$. If $k|n$, $1 < k < n$ then $x^{n/k}$ has order $1 < k < n$ which is impossible. Thus n is prime.

(3) Take $a \in A$ then $(A \cap H)^a = A$ since A is abelian. Take $h \in H$ then $(A \cap H)^h \subseteq H$ (always) and $(A \cap H)^h \subseteq A$ since $A \lhd G$. So $(A \cap H)^h \subseteq A \cap H$. Since $AH = G$, $A \cap H \lhd G$.

(4) (a) Let $N = \ker(\phi)$. Then $N \lhd G$ so NH is a subgroup of G. So by the correspondence theorem $|G : NH| = |\phi(G) : \phi(H)|$. Also $|G : NH|$ divides $|G : H|$ since $H \subseteq NH$. Thus $|\phi(G) : \phi(H)|$ contains no primes from π.
Also by the isomorphism theorems $\phi(H) \cong H/(H \cap N)$ so the order of $\phi(H)$ divides the order of H.
Thus $\phi(H)$ is a Hall π-subgroup.

(b) Suppose $G = HN$. Then $\phi(G) \cong G/N = HN/N \cong H/(H \cap N)$ which is a π-group.
Suppose $\phi(G)$ is a π-group. Then $\phi(G) \cong G/H$ so $|H|$ contains the full powers that appear in $|G|$ of every prime not in π. Also $|HN| = |H||N|/|H \cap N|$ so HN contains the full powers that appear in $|G|$ of every prime not in π. Thus $|G : HN|$ is only divisible by primes in π. But $|G : HN|$ divides $|G : H|$ and so contains only primes not in π. Thus $|G : HN| = 1$.

(5) A is cyclic since it is of order p. Let a be a generator. Take any $x \in P$. Then $x^{-1}ax = a^k$ for some $1 \leq k \leq p - 1$ since $A \lhd P$. Thus, as in an example from class, $a = xa^kx^{-1} = a^{k^2}$. So $p|k^2 - 1 = (k + 1)(k - 1)$. Due to the restrictions on k this means $k = 1$ or $k = -1$.

Thus P has been partitioned into two sets – those elements, P_+, which lead to $k = 1$ and those, P_-, which lead to $k = -1$. The k of a product of two elements is the product of the ks. If $P_- = \varnothing$ then we're done. Suppose otherwise that $z \in P_-$. Then we have a set bijection

$$P_+ \to P_-$$
$$y \mapsto yz$$

So $|P| = 2|P_+|$ so the order of P is a power of 2. Thus the order of A is 2 so $a = a^{-1}$ and so in all cases the elements of P commute with a.

(6) (a) From class we know that if $P \subseteq N_G(Q)$ then $P \subseteq Q$. Now, $P \subseteq N_G(P) = N_G(Q)$ so $P \subseteq Q$. By symmetry (or order) then $P = Q$.

(b) $P \in Syl_p(N_G(N_G(P)))$ Any conjugate Q of P by an element of $N_G(N_G(P))$ is in $N_G(P)$ since $P \subseteq N_G(P)$ and $N_G(P) \lhd N_G(N_G(P))$. Thus, as in the previous part, $Q = P$. So P is the unique Sylow-p-subgroup of $N_G(N_G(P))$. Thus

$P \lhd N_G(N_G(P))$. But $N_G(P)$ in the maximal subgroup in which P is normal. So $N_G(P) = N_G(N_G(P))$.

(7) Let $|G| = 105$. $105 = 7 \cdot 5 \cdot 3$, $35 = 7 \cdot 5$.

$n_7 | 5 \cdot 3$ and $n_7 \equiv 1 \mod 7$ so $n_7 = 1$ or $n_7 = 15$. $n_5 | 7 \cdot 3$ and $n_5 \equiv 1 \mod 5$ so $n_5 = 1$ or $n_5 = 21$.

Suppose $n_7 = 15$ and $n_5 = 21$. Then, since cyclic groups of prime order have no nontrivial proper subgroups, G would have $15 \cdot 6$ elements of order 7 and $21 \cdot 5$ elements of order 5 giving 174 elements. A contradiction.

So let $P \in \mathrm{Syl}_7(G)$ and $Q \in \mathrm{Syl}_5(G)$. Then either $P \lhd G$ or $Q \lhd G$ (or both), so PQ is a subgroup of G and $|PQ| = |P||Q|/|P \cap Q| = 35$.