HOMEWORK 3 SOLUTIONS

MATH 818, FALL 2010

Sh, I.6.6: Let $\overline{X} \in \mathbb{P}^3$ be the projective closure of X. Let $\widetilde{X}$ be any component of $\overline{X}$. Project away from where the z axis meets the plane at infinity. We know the projection map

$$\pi : \widetilde{X} \to \mathbb{P}^2$$

is regular, and hence $\pi(\widetilde{X})$ is closed. By dimension $\pi(\widetilde{X})$ has no component of dimension 2 (if it did it would be onto and then I.6.3 Theorem 7 would give the contradiction).

Suppose $\pi(\widetilde{X})$ had a component of dimension 0, that is a point $P \in \mathbb{P}^2$ which has a neighbourhood $U \subset \mathbb{P}^2$ such that $\pi^{-1}(P) = \pi^{-1}(U)$. Let $V = U \times \mathbb{P}^1$ (where the $\mathbb{P}^1$ has coordinate z); then $\pi^{-1}(U) = \overline{X} \cap (V)$ consists only of points on the line parallel to the z axis containing $(P, 0)$, which is a contradiction because X does not contain any lines parallel to the z axis, and is closed and pure dimension 1.

Thus $\pi(\widetilde{X})$ is pure codimension 1. Now restrict back to affine space and apply Theorem 3, I.6.1 to get that $Y = \pi(\widetilde{X}) \cap \mathbb{A}^2$ is a hypersurface and its maximal ideal is principal. Finally note that Y is exactly the closure of the projection of X parallel to the z-axis and that any polynomial which vanishes on Y is a polynomial in x, y which also vanishes on any point of X.

Sh, I.6.7: Write

$$f(x, y, z) = e_0(x, y)z^m + \cdots + e_m(x, y)$$

Then

$$\begin{aligned} fg_0^m &= h(x,y,z)g_0^{m-1}e_0z^{m-n} + g_0^{m-1}z^{m-1}(e_1g_0 - e_0g_1) \\ &\quad + g_0^{m-1}(\text{stuff of degree at most } m-2 \text{ in } z) \\ &= h(x,y,z)(g_0^{m-1}e_0z^{m-n} + g_0^{m-2}(e_1g_0 - e_0g_1)) + g_0^{m-2}z^{m-2}(\text{a constant}) \\ &\quad + g_0^{m-2}(\text{stuff of degree at most } m-3 \text{ in } z) \end{aligned}$$

Continuing likewise get

$$f(x, y, z)g_0^m(x, y) = h(x, y, z)U(x, y, z) + v(x, y, z)$$

where v is of degree at most $n-1$ in z. But $f(x, y, z)g_0^m(x, y)$ is zero on X and so by the minimality of g, v depends only on x and y. Thus if $g(x, y)$ is the generator of the ideal of Y (from the previous question), then g divides v.

Then $V(h, g) = V(\mathfrak{A}_X, g_0, g)$ defines a curve consisting of X along with finitely many lines parallel to the z axis defined by $g_0 = g = 0$.

Sh, I.6.8: Pick a point P_i on each line L_i in $V(g_0, g)$ which is not in X. Choose an element k of $\mathfrak{A}_X$ which is nonzero on each P_i. This is possible, as we can certainly choose an element f_i of $\mathfrak{A}_X$ which is nonzero on P_i and then some linear combinationo of the

f_i will do. Then $X \subset V(h,g,k)$, and $V(h,g,k) \subset V(h,g) = X \cup \bigcup L_i$. But no P_i is in $V(h,g,k)$ and $V(h,g,k)$ is closed in $X \cup \bigcup L_i$ so $V(g,h,k) = X$.

Sh, II.1.1: Take any $f \in \mathcal{O}_x$ then f is regular on some neighbourhood of x so $f \in k[U]$ for some U a neighbourhood of x. Now take $f \in \bigcup k[U]$; this says that f is regular on some neighbourhood of x and in particular then it is regular at x and so is in $\mathcal{O}_x$.

F, 7-2: *Note that I don't care if you just blowup the singularity (and take the strict transform) or if you then project onto the x, z plane to get a plane curve back. The latter is what Fulton calls F' but we didn't discuss that and either is fine.*

(a) Consider each example in turn

$Y - X^2$: Substitute $y = x^2$ into $y - xz$ to get $x^2 - xz = x(x - z)$, and so the exceptional divisor is given by $x = 0, y = 0$, and $F' = V(y = xz, x = z)$. To check that this is nonsingular calculate the Jacobian matrix

$$\begin{bmatrix} -z & -1 \\ 1 & 0 \\ -x & 1 \end{bmatrix}$$

and notice that this matrix has rank 2 for all values of x, y, z.

$Y^2 - X^3 + X$: Substitute to get $x^3 - x - x^2z^2 = x(x^2 - 1 - xz^2)$, so $F' = V(y = xz, x^2 = 1 + xz^2)$. The Jacobian matrix is

$$\begin{bmatrix} -z & 2x \\ 1 & 0 \\ -x & -2xz \end{bmatrix}$$

which has rank 2 on the curve since $2x = 0$ does not satisfy the second equation.

$Y^2 - X^3$: Following the same steps we have $x^3 - x^2z^2 = x^2(x - z^2)$ so $F' = V(y = xz, x = z^2)$ and

$$\begin{bmatrix} -z & 1 \\ 1 & 0 \\ -x & 2z \end{bmatrix}$$

has rank 2.

$Y^2 - X^3 - X^2$: $F' = V(y = xz, x + 1 = z^2)$ and

$$\begin{bmatrix} -z & 1 \\ 1 & 0 \\ -x & -2z \end{bmatrix}$$

has rank 2.

$(X^2 + Y^2)^2 + 3X^2Y - Y^3$: Substitute $(x^2 + x^2z^2)^2 + 3x^3z - x^3z^3 = x^3(x + 2xz^2 + xz^4 + 3z - z^3)$ so $F' = V(y = xz, x + 2xz^2 + xz^4 + 3z - z^3 = 0)$ which has Jacobian matrix

$$\begin{bmatrix} -z & 1 + 2z^2 \\ 1 & 0 \\ -x & 4xz + 4xz^3 + 3 - 3z^2 \end{bmatrix}$$

To check the rank of this note that if $1+2z^2 = 0$ and $4xz+4xz^3+3-3z^2 = 0$ then we do not get a point on the curve. Thus there are no points on the curve where the rank of the Jacobian is other than 2.

$(X^2+Y^2)^3-4X^2Y^2$: As in the previous part get $(x^2+x^2z^2)^3-4x^4z^2 = x^4(x^2(1+z^2)^3-4z^2)$. So $F' = V(y = xz, x^2(1+z^2)^3 = 4z^2)$. The Jacobian matrix is

$$\begin{bmatrix} -z & 2x(1+z^2)^3 \\ 1 & 0 \\ -x & 6z(1+z^2)^2-8z \end{bmatrix}$$

This one finally is still singular at $(0,0,0)$.

(b) One more time on Y^2-X^5 we get $x^2z^2-x^5 = x^2(z^2-x^3)$ so $F' = V(y = xz, z^2 = x^3)$. The Jacobian is

$$\begin{bmatrix} -z & -3x^2 \\ 1 & 0 \\ -x & 2z \end{bmatrix}$$

This is singular at $(0,0,0)$ and nowhere else, but let's blow up again. This time we are blowing up $(0,0,0)$ in A^3. Considering blowing up all of A^3 with coordinates (x,y,z) and with the two new blowup coordinates being (t,u) this gives the variety $V(y = xt, z = xu)$. Now consider what happens to F'. Substituting we get $0 = xz-xt = x(z-t)$ and $x^2u^2-x^3 = x^2(u^2-x)$, so the exceptional divisor is $V(x,y,z)$ and $F'' = V(z = t, u^2 = x, y = xt, z = xu)$ which has Jacobian

$$\begin{bmatrix} 0 & 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 0 & 2u \\ -t & 1 & 0 & -x & 0 \\ -u & 0 & 1 & 0 & -x \end{bmatrix}$$

which has rank 4 for all points on F'' and hence F'' is smooth. So for Y^2-X^{2n+1} we should expect to need to blow up n times.

Sh, II.4.4: ϕ is certainly rational. The inverse map is

$$(y_0 : y_1 : y_2 : y_3 : y_4) \mapsto (y_0 : y_2 : y_3)$$

and so ϕ is birational to $\overline{\phi(\mathbb{P}^2)}$. Moreover the blowup of $\mathbb{P}^2$ at $(1:0:0)$ is the variety in $\mathbb{P}^2 \times \mathbb{P}^1$ given by $V(x_1'y_2' = x_2'y_1')$ where the coordinates of the $\mathbb{P}^2$ are x_0', x_1', x_2' and the coordinates of the $\mathbb{P}^1$ are y_1', y_2'. If we label the coordinates of the image of ϕ by $(x_0' : y_1' : x_1' : x_2' : y_2')$, then we get

$$x_0' = x_0x_1 \quad x_1' = x_1^2 \quad x_2' = x_1x_2 \quad y_1' = x_0x_2 \quad y_2' = x_2^2$$

which satisfies $x_1'y_2' = x_2'y_1'$. Furthermore if at least one of x_0' or x_2' is nonzero (call this set U) then $x_1 \neq 0$ so dividing by x_1 we see that ϕ is an isomorphism; on the other hand if $x_0' = x_2' = 0$ then we get the points $(0:0:0:x_0:1)$ and $(0:1:0:0:0)$ in the image of ϕ all of which are in the closure of $\phi(U)$. Hence the inverse of ϕ is the blowup.

Sh, III.1.2: The function is $(x_1-x_0)/x_0$. Consider $\operatorname{div}(x_1-x_0)$. The only point of intersection is $(1:1:0)$ and so by Bezout we know the intersection multiplicity is 2. So $\operatorname{div}(x_1-x_0) = 2(1:1:0)$.

On the other hand the points of intersection of $x_0 = 0$ with the curve are $(0:1:i)$ and $(0:1:-i)$. Again by Bezout we know that each of these intersections has multiplicity 1. So $\mathrm{div}(x_0) = (0:1:i) + (0:1:-i)$ and so $\mathrm{div}((x_1 - x_0)/x_0) = 2(1:1:0) - (0:1:i) - (0:1:-i)$.

Sh, III.1.5: Suppose $k[X]$ is a UFD.Consider the proof of I.6.1 Theorem 3. The only special property of $k[\mathbb{A}^n]$ needed in this proof was that this ring is a UFD (we observed this fact when proving II.3.1 Theorem 1 which was the same argument applied to $\mathcal{O}_x$ at a smooth point). Now one more time apply this argument this time using that $k[X]$ is a UFD. Conclude that every pure codimension 1 subvariety is defined by a single equation and its ideal is principal. Hence, for $C = V(F)$ we get that $\mathrm{div}F = C$ and thus every prime divisior and so every divisor is principal.

Now suppose $\mathrm{Cl}X = 0$. Then in particular every prime divisor is principal, so for every irreducible codimension 1 subvariety C there is an $f \in k(X)$ such that $\mathrm{div}(f) = C$. But then $\mathrm{div}(f) > 0$ and so $f \in k[X]$. Now suppose $k[X]$ does not have unique factorization; so there exists $f, g, h \in k[X]$ irreducible such that $f | gh$ but $f \not| g$, $f \not| h$. So we have $\mathrm{div}(gh/f) = V(g) + V(h) - V(f)$ but we also have $gh/f \in k[X]$ so $\mathrm{div}(gh/f) > 0$ which is a contradiction.

Sh, III.1.12: X has one singularity: $(0:0:1)$. Take any locally principal divisor D of X. By moving the support away from $(0:0:1)$ we obtain a divisor D' with $D' \sim D$ and with $(0:0:1) \notin \mathrm{Supp}D'$. But on the smooth part of X we have all the results on divisors on smooth curves. In particular we have a well defined notion of degree on divisor classes on $X \setminus (0:0:1)$. Suppose D_1 and D_2 were locally principal divisors of X and suppose D_1' and D_2' were divisors on $X \setminus (0:0:1)$ with

$$D_1 \sim D_1' \quad D_2 \sim D_2' \quad D_1 \sim D_2.$$

Then $D_1' \sim D_2'$ on X so there exists $f \in k(X)$ such that $D_1' = D_2' + \mathrm{div}(f)$. But $(0:0:1)$ is not in the support of either D_1' or D_2' so $f \in k(X \setminus (0:0:1))$ with the same divisor. Thus $\deg(D_1') = \deg(D_2')$ and so the degree is well defined on locally principal divisors of X.

Thus we have

$$\deg : \mathrm{Pic}X \to \mathbb{Z}$$

and this map is onto. The kernel is $\mathrm{Pic}^0 X$, the locally principal divisors of degree 0, and so $\mathrm{Pic}X \cong \mathbb{Z} \oplus \mathrm{Pic}^0 X$. Now pick any $\alpha_0 \in X \setminus (0:0:1)$. Consider

$$\begin{aligned} \phi : X \setminus (0:0:1) &\to \mathrm{Pic}^0 X \\ P &\mapsto P - \alpha_0 \end{aligned}$$

Consider the constructions of addition and negation for elliptic curves. This curve is also cubic so again each line will intersect it in three points. Further $(0:0:1)$ is a singularity of multiplicity 2 and so if two points on a line are determined (with multiplicity) then $(0:0:1)$ can never be the third point. Thus the same construction as for elliptic curves gives that ϕ is a bijection.

Finally now parametrize $X \setminus (0:0:1)$ via $(t^2 : t^3 : 1)$. Note that $(0:1:0)$ is a flex of this curve and so use it as α_0. Thus as for elliptic curves (by the same calculations) we get

$$\ominus(t^2 : t^3 : 1) = (t^2 : -t^3 : 1) = ((-t)^2 : (-t)^3 : 1)$$

and for $t_1 + t_2 \neq 0$ the x coordinate of $(t_1^2 : t_1^3 : 1) \oplus (t_2^2 : t_2^3 : 1)$ is

$$\left(\frac{t_2^3 - t_1^3}{t_2^2 - t_1^2}\right)^2 - t_1^2 - t_2^2 = \left(\frac{t_2 t_1}{t_2 + t_1}\right)^2$$

and so the y coordinate is

$$-\left(t_1^3 + \frac{t_2^3 - t_1^3}{t_2^2 - t_1^2}\left(\left(\frac{t_2 t_1}{t_2 + t_1}\right)^2 - t_1^2\right)\right) = \left(\frac{t_2 t_1}{t_2 + t_1}\right)^3$$

Now consider the change of variables $u_i = 1/t_i$. Considered in the u variables negation remains usual negation and addition becomes usual addition. The point $(0 : 0 : 1)$ is now at infinity and the point $(0 : 1 : 0)$ is $u = 0$, so the parameter runs over k to capture the points we are interested in. Thus $\mathrm{Pic}^0 X$ is isomorphic to k with usual addition and negation and so

$$\mathrm{Pic} X \cong \mathbb{Z} \oplus k$$

Sh, III.1.18: Consider any divisor D of X. Restricting D to $\sigma^{-1}(y) \cong y \times \mathbb{P}^{n-1}$ we get a divisor of $\mathbb{P}^{n-1}$ and linearly equivalent divisors on X give linearly equivalent divisors on $\mathbb{P}^{n-1}$. Thus define $\deg(D) = \deg(D|_{\sigma^{-1}(y)})$. Note that any prime divisor which is not disjoint from $\sigma^{-1}(y)$ has nonzero degree, so by adding and subtracting divisors we get that degree is surjective on $\mathbb{Z}$. Thus we have

$$\deg : \mathrm{Cl} X \to \mathbb{Z}.$$

Consider now the kernel, $\mathrm{Cl}^0 X$, of the degree map. This is isomorphic to $\mathrm{Cl} Y$ since any divisor of degree 0 on X is linearly equivalent to one which is not supported on $\sigma^{-1}(y)$, among such divisors linear equivalence in Y is the same as linear equivalence in X, and any divisor of Y is linearly equivalent to one not supported at y.