HOMEWORK 4 SOLUTIONS

MATH 818, FALL 2010

Comparison: The proofs have some similarities and some differences so both answers are possible. Here are a few comments:

Both proofs are fundamentally about intersection numbers; Fulton has spent some time developing the theory of intersection numbers for curves while for Shafarevich they are a new definition, but Shafarevich defines them in terms of the degree of a divisor (the theory of which has been previously developed), whereas only in a later chapter does the connection with divisors become clear in Fulton. Whether this aspect of the proofs all and all is more similar or more different depends on the scale we view it at.

Fulton reduces to the affine case and does fundamentally affine calculations (with some exact sequences of vector spaces of homogeneous polynomials). Shafarevich does his main calculations locally on the tangent spaces. First note that the difference between calculations on tangent spaces and on local rings really is a difference of set up more than a fundamental difference. Fulton has already developed a local theory of intersection numbers (the "Chinese remainder-type theorem" which came up again and again is the most striking example as it tells us how to glue the local properties into a global affine statement), so again how much this difference of affine vs. local calculations is a minor difference of language or a fundamental difference really depends on the scale at which we're viewing the proof.

Shafarevich proves a more general result intersecting a curve in any dimension with a hypersurface while Fulton only works in dimension two, that is only intersects pairs of curves. This is a consequence of the scope of Fulton's book. Finally Shafarevich uses properties of divisors to reduce to intersections of curves with linear hypersurfaces. This is in some sense analogous to part of the exactness of Fulton's sequences, but I think it is fair to call this a fundamental difference between the proofs.

F, 8-10: Put C is Weierstrass form (as in question 8-2). If $r = 0$ we get k and there's nothing else to say. Suppose $r > 0$. Calculate $(z)_0$. $Z = 0$ intersects C only at the point at infinity, with multiplicity 3. $X = 0$ intersects C at $(0 : 0 : 1)$ with multiplicity 2 and at $(0 : 1 : 0)$ with multiplicity 1 and so $(z)_0 = 2(0 : 1 : 0)$.

Suppose $f \in \mathcal{L}(r(z)_0)$. Then $2r(0 : 1 : 0) + \mathrm{div}(f) > 0$. Dehomogenize f with $z \neq 0$ giving $f(x, y) = G/H$, $G, H \in k[x, y]$. But $f(x, y)$ has no poles (since its only possible pole was the one we removed by dehomogenizing). Thus $f \in k[x, y]$.

Now calculate dimensions. First note that using the fact that C is in Weierstrass form we can represent f so uniquely with degree at most 1 in y by cacncelling off higher powers of y. So we just need to check at which value of r each monomial appears. Let $M = x^a y^b$. Let α, β, γ be the three roots when $Y = 0$, $Z = 1$. Then

$$\begin{aligned}\operatorname{div}(M) &= a(0:1:0) + 2a(0:0:1) + b(\alpha:0:1) + b(\beta:0:1) + b(\gamma:0:1) - 3(a+b)(0:1:0) \\ &= 2a(0:0:1) + b(\alpha:0:1) + b(\beta:0:1) + b(\gamma:0:1) - (2a+3b)(0:1:0)\end{aligned}$$

So

$$\ell(r(z)_0) = |\{(a,b) : a \in \mathbb{Z}_{\geq 0}, b \in \{0,1\}, 2a + 3b \leq 2r\}|$$

which we can prove is $2r$ by an easy induction.

F, 8-14(a,b): (a) The partial derivatives of $X^2Y^2 - Z^2(X^2+Y^2)$ are

$$\begin{aligned}&2XY^2 - 2XZ^2 \\ &2YX^2 - 2YZ^2 \\ &-2Z(X^2+Y^2)\end{aligned}$$

setting these each equal to 0 we get

$$\begin{aligned}(X = 0 \text{ or } Y = Z \text{ or } Y = -Z) \text{ and} \\ (Y = 0 \text{ or } X = Z \text{ or } X = -Z) \text{ and} \\ (Z = 0 \text{ or } X = iY \text{ or } X = -iY)\end{aligned}$$

The only solutions to these are $(0:0:1)$, $(0:1:0)$, and $(1:0:0)$ which are all on the curve. To find the multiplicities just set $Z = 1$, $Y = 1$, $X = 1$ respectively and take the lowest degree appearing. In each case this is 2. So using the formula for genus we get

$$g = \frac{3 \cdot 2}{2} - 3\left(\frac{2 \cdot 1}{2}\right) = 0$$

(b) The partial derivatives of $(X^3+Y^3)Z^2 + X^3Y^2 - X^2Y^3$ are

$$\begin{aligned}&3X^2Z^2 + 3X^2Y^2 - 2XY^3 \\ &3Y^2Z^2 + 2X^3Y - 3X^2Y^2 \\ &-2Z(X^3+Y^3)\end{aligned}$$

setting these each equal to 0 we get

$$\begin{aligned}(X = 0 \text{ or } 3XZ^2 + 3XY^2 = 2Y^3) \text{ and} \\ (Y = 0 \text{ or } 3YZ^2 + 2X^3 = 3X^2Y) \text{ and} \\ (Z = 0 \text{ or } X = \xi Y \text{ for } \xi^3 = -1)\end{aligned}$$

First suppose $Z = 0$, then $X = 0$ or $3XY^2 = 2Y^3$, so $X = 0$ or $Y = 0$ or $3X = 2Y$. But then by the second equation $Y = 0$ or $X = 0$ or $2X = 3Y$. The only projective points satisfying these restrictions are $(1:0:0)$ and $(0:1:0)$. Now suppose $Z \neq 0$. Note that $(0:0:1)$ is a solution. Otherwise let $X = 1$, $Y = \xi$. Then from the first equations

$$3Z^2 = -2 - 3\xi^2 \quad 3\xi Z^2 = 3\xi - 2$$

Solving both equations for $3Z^2$ and equating we get

$$-2 - 3\xi^2 = 2\xi^2 + 3$$

so $\xi^2 = -1$ which is a contradiction. Thus we have only three singularities.
To find the multiplicities again just set $Z = 1$, $Y = 1$, $X = 1$ and take the lowest degree appearing. In each case we get $3, 2, 2$. So using the formula for genus we get

$$g = \frac{4 \cdot 3}{2} - \frac{3 \cdot 2}{2} - 2\left(\frac{2 \cdot 1}{2}\right) = 6 - 3 - 2 = 1.$$

Sh, III.3.1: Suppose $\mathrm{char}(k) \neq 2, 3$. Suppose P has order 2. That is, $2P = 0 = -0$, so the tangent line at P goes through 0. In Weierstrass form this means that the tangent at P is vertical. Furthermore if the tangent is vertical then $2P = 0$ so P has order 2. Let the curve be defined by

$$y^2 = x^3 + ax + b$$

So

$$2y = 3x^2 \frac{dx}{dy} + a\frac{dx}{dy}$$

Thus if the tangent is vertical then we must have $y = 0$. So the points of order 2 are among the roots of $x^3 + ax + b$. By assumption the curve is smooth and so these are distinct. Now suppose $P = (x_0, 0)$ is on the curve. Then $0 = (3x_0^2 + a)\frac{dx}{dy}(x_0, 0)$ and $0 = x_0^3 + ax_0 + b$. But since $x^3 + ax + b$ has no repeated roots, then it has no roots in common with $3x^2 + a$, and so we must have $\frac{dx}{dy}(P) = 0$ giving that P is a point of order 2.