Math 821, Spring 2013, Lecture 12

Karen Yeats
(Scribe: Yue Zhao)

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1 Duals

Definition. (1) *Let V be a finite dimensional vector space over k, then $V^* = Hom(V, k)$ the space of linear maps from V to k.*

(2) *If $\Phi : V \to W$ is a linear map, then $\Phi^* : W^* \to V^*$ by $(\Phi^*(f))(v) = f(\Phi(v))$.*

(3) *If $v_1, v_2, \cdots, v_n$ is a basis for V, let $f_i \in V^*$ be the map $f_i(v_i) = 1$, $f_i(v_j) = 0$ for $i \neq j$. Then $f_1, f_2, \cdots, f_n$ is a basis of V^*, called the dual basis.*

Note. *(2) says taking duals reverses arrows. So we should expect algebra$\to$ coalgebra and vice versa.*

Definition. *A graded $k-$vector space $V = \bigoplus_{i=0}^{\infty} V_i$ is of finite type if each V_i is finite dimensional. Note if $\mathcal{C}$ is a combinatorial class then $V\mathcal{C} = \bigoplus_{n=0}^{\infty} V\mathcal{C}_n$ is of finite type.*

Definition. *Let $V = \bigoplus_{n=0}^{\infty} V_n$ be a graded vector space of finite type. Then the* restricted dual *is $V^o = \bigoplus_{n=0}^{\infty} V_n^*$.*

Note. *The elements of V^o are linear maps from $V \to k$ which vanish on all but finitely many of the V_n.*

So if we have a graded algebra, take its restricted dual (from now on just call this the dual) get a coalgebra and vice versa. So a graded bialgebra will have a dual which is also a graded bialgebra.

Note. *Connected is preserved under duals because $k^* = Hom(k, k) \cong k$. So the dual of a graded connected finite type Hopf algebra is a graded connected finite type Hopf algebra.*

But what does this look like concretely?

Let A be a graded connected finite type Hopf algebra. What is $\triangle_{A^o}$?

$$\triangle_{A^o} : A^o \to A^o \otimes A^o$$
$$\triangle_{A^o}(f)(a \otimes b) = f(ab)$$

where $A^o = \bigoplus_{n=0}^{\infty} Hom(A_n, k) \subseteq Hom(A, k)$. What is $\cdot_{A^o}$?

$$\cdot_{A^o} : A^o \otimes A^o \to A^o$$
$$(\cdot_{A^o}(f \otimes g))(a) = (f \otimes g)(\triangle(a))$$

Write this in terms of a basis.

Proposition 1. *Say $\{a_i\}_{i\in I}$ a basis for A a graded connected finite type Hopf algebra and let $\{f_i\}_{i\in I}$ be the dual basis, write*

$$a_j a_k = \sum_{i\in I} c^i_{j,k} a_i$$
$$\text{then } \triangle_{A^o}(f_i) = \sum_{(j,k)\in I\times I} c^i_{j,k} f_j \otimes f_k$$

and dually, write

$$\triangle(a_i) = \sum_{(j,k)\in I\times I} d^i_{j,k} a_j \otimes a_k$$
$$\text{then } f_j \cdot_{A^o} f_k = \sum_{i\in I} d^i_{j,k} f_i$$

Proof. It suffices to prove the first part by the previous observation.

$$\triangle_{A^o}(f_i)(a_j \otimes a_k) = f_i(a_j a_k) = f_i(\sum_{l\in I} c^l_{j,k} a_l) = \sum_{l\in I} c^l_{j,k} f_i(a_l) = c^i_{j,k}$$

$\square$

Example. *Let TV =words let $A = TV$, what is A^o?*

A basis for A is given by words, so the dual basis for A^o is indexed by words, thinking of a basis element as its index, we can view A^o as also being a Hopf algebra of words.

Recall multiplication on A is concatenation and comultiplication on A is anti-shuffle. So comultiplication of A^o is deconcatenation

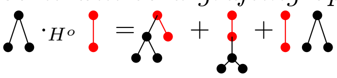
$$\triangle_{A^o}(abcd) = \mathbb{1} \otimes abcd + a \otimes bcd + ab \otimes cd + abc \otimes d + abcd \otimes \mathbb{1}$$

multiplication on A^o is shuffle

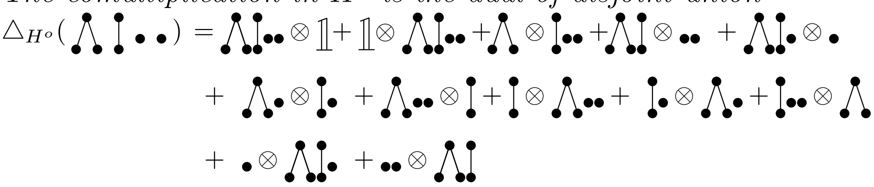
$$ab \cdot_{A^o} cd = abcd + acbd + cabd + acdb + cadb + cdab$$

Example. *Let H be the Connes-Kreimer Hopf algebra of rooted trees, what is H^o?*

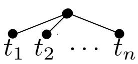
What is multiplication in H^o, it has to be dual to taking admissible cuts so it will be a grafting operator. eg.



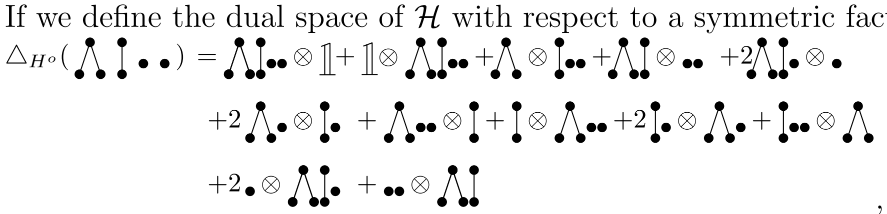
The comultiplication in H^o is the dual of disjoint union



I partition the connected components into two parts in all possible ways.

This is almost isomorphic to the Grossman-Larson Hopf algebra in the following way for each forrest $t_1, t_2, \cdots, t_n$ of H^o form the tree , these are the basis for the Grossman-Larson Hopf algebra G (note no empty tree here). Now the size of a tree is its number of edges, so $\bullet$ is size 0.

If we define the dual space of $\mathcal{H}$ with respect to a symmetric factor, i.e,



We can import $\cdot_{H^o}$, $\triangle_{H^o}$. $t_1 \cdot_G t_2$ grafts the child of the root of t_1 into t_2 (actually usually define the opposite of this, i.e. graft children of t_2 into t_1) and $\triangle_G(t)$ partitions the children of root of t onto the two sides of $\otimes$ and given them a new root on each side.

References. *Victor Reiner, Hopf Algebras In Combinatorics, Cha 1.*
For Grossman-Larson Hopf algebra: see arXiv:math/0003074, Florin Panaite, Relating the Connes-Kreimer and Grossman-Larson Hopf algebras built on rooted trees.
arXiv:math/0201253, Michael E. Hoffman, Combinatorics of Rooted Trees and Hopf Algebras.

2 B_+

Working in the Connes-Kreimer Hopf algebra H, what are the primitive elements?

$\bullet$, $\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix} - \frac{1}{2}\bullet\,\bullet$

(recall primitive $\triangle(a) = a \otimes \mathbb{1} + \mathbb{1} \otimes a$)

Check $\tilde{\triangle}(\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix} - \frac{1}{2}\bullet\,\bullet) = \bullet \otimes \bullet - \frac{1}{2}(\bullet \otimes \bullet + \bullet \otimes \bullet) = 0$

Definition. *$B_+ : H \to H$ is the linear function which takes a forest $t_1, t_2, \cdots, t_n$ and returns* $\overset{\bullet}{\overbrace{t_1\; t_2\; \cdots\; t_n}}$

What is $\triangle B_+$?

$$\begin{aligned}
\triangle B_+(t_1 t_1 \cdots t_n) &= \triangle(\overset{\bullet}{\overbrace{t_1\; t_2\; \cdots\; t_n}}) \\
&= \overset{\bullet}{\overbrace{t_1\; t_2\; \cdots\; t_n}} \otimes \mathbb{1} + (id \otimes B_+) \prod_{i=1}^{n} \sum_{\substack{c \\ admissible\ cut\ of\ t_i}} P_c(t_i) \otimes R_c(t_i) \\
&= B_+(t_1 t_1 \cdots t_n) \otimes \mathbb{1} + (id \otimes B_+)(\triangle(t_1 t_1 \cdots t_n)) \\
\Rightarrow \triangle B_+ &= B_+ \otimes \mathbb{1} + (id \otimes B_+)\triangle
\end{aligned}$$

3 3 line summary of cohomology

- You need a family maps b_n from objects of size n to objects of size $n+1$ with $b^2 = b_{n+1} b_n = 0$.

- Take quotients $Ker(b)/Im(b)$.

- Use these to understand your original objects.

For us we want "objects of size n" to be $Hom(H, H^{\otimes n})$ (actually H could be any bialgebra here) and $b : Hom(H, H^{\otimes n}) \to Hom(H, H^{\otimes n+1})$.

$$bL = (id \otimes L)\triangle + \sum_{i=1}^{n} (-1)^i \triangle_i L + (-1)^{n+1} L \otimes \mathbb{1}$$
$$where,\ \triangle_i = id \otimes \cdots \otimes \triangle \otimes id \cdots \otimes id,\ \triangle\ is\ the\ i-th\ part.$$

This gives the Hochschild cohomology of bialgebras. If I were to do this, one of the first thing I'd need to know is $Ker(b_1)$

$$\begin{aligned} 0 &= b_1 L \\ &= (id \otimes L)\triangle - \triangle L + L \otimes \mathbb{1} \\ so\ \triangle L &= L \otimes \mathbb{1} + (id \otimes L)\triangle \end{aligned}$$

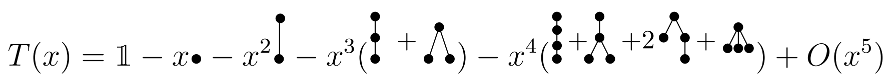
that's the property B_+ has, this is called the 1-cocycle property.

Comment. *This 1-cocycle property is really important in these Renormalization Hopf algebras (like Connes-Kreimer) but I haven't seen it appear in other combinatorial hopf algebras.*

4 Specifications and Combinatorial Dyson-Schwinger equations

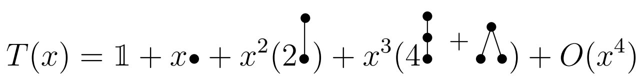
Again let H be the Connes-Kreimer Hopf algebra. I can use B_+ to give combinatorial specifications in a different languages.

Example. $T(x) = \mathbb{1} - xB_+(\frac{1}{T(x)})$
I want a solution to this in $H[[x]]$, expand this recursively



this is just $\mathcal{T} = \mathcal{E} + \mathcal{Z} \times Seq(\mathcal{T} - \mathcal{E})$, plane rooted trees then forget the plane structure giving above coefficients.

Example. $T(x) = \mathbb{1} + xB_+(T(x)^2)$



$\mathcal{T} = \mathcal{E} + \mathcal{Z} \times \mathcal{T}^2$ *and forget the extra structure to get the coefficients.*

These equations (and more general ones) are called combinatorial Dyson-Schwinger equations (this name is given by Dr. Karen Yeats) they give some specifications in the Hopf algebra context. They are physically important and give the sums of trees/ feynman graphs which contributes to a given physical process.