Recall from Lecture 6:

**Proposition 1.** Suppose $T(x) \in \mathbb{R}_{\geq 0}[[x]]$, $E(x, y) \in \mathbb{R}_{\geq 0}[[x, y]]$ with

- $E(0, 0) = 0$,
- $E$ has a term of degree $> 1$ in $y$,
- $\frac{d}{dx} E(x, T(x)) \neq 0$ (so since coefficients are nonnegative, in particular $\frac{d}{dx} E(\rho, T(\rho)) \neq 0$)

and $T(x) = E(x, T(x))$, as formal power series. Let $\rho$ be the radius of convergence of $T(x)$ and suppose $0 < \rho < \infty$, $T(\rho) < \infty$ and $\exists \epsilon$ such that $E(\rho + \epsilon, T(\rho) + \epsilon) < \infty$. Then $\exists$ functions $A(x), B(x)$ analytic at 0 such that

$$T(x) = A(\rho - x) + B(\rho - x) \sqrt{\rho - x}$$

for $|x| < \rho$, $x$ near $\rho$.

### 1 Proof of the square root result

**Theorem 2** (Weierstraß preparation). Let $f : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and let $f$ be analytic in a neighbourhood of $(0,0)$. Suppose

$$f(0, 0) = \frac{d}{dy} f(0, 0) = \cdots = \frac{d^{k-1}}{dy^{k-1}} f(0, 0) = 0, \quad \text{but} \quad \frac{d^k}{dy^k} f(0, 0) \neq 0.$$

Then in a neighbourhood of $(0,0)$ we can uniquely write $f(x, y) = p(x, y)r(x, y)$ where

- $p, r$ analytic in the neighbourhood
- $r$ is nowhere 0 in the neighbourhood
- $p(x, y) = p_0(x) + p_1(x)y + \cdots + p_{k-1}y^{k-1} + y^k$ (a Weierstraß polynomial) with the $p_i$ analytic in a neighbourhood of 0 and $p_i(0) = 0$
Sketch of proof. (For details see analysis text.)
Unique by expanding series.
By conditions on $f$,

- $\frac{d^k}{dy^k} f(x, y)$ is nonzero at $(0, 0)$, so there exists a small neighbourhood of $(0, 0)$ where it is nowhere 0,

- $f(0, y)$ has a root at 0 of multiplicity $k$, so for fixed $x_0$ sufficiently near 0, $f(x_0, y)$ has $k$ roots (maybe distinct)

So there exists a Weierstraß polynomial with the same root structure, call it $p(x, y)$. Then

$$\frac{f(x, y)}{p(x, y)}$$

is analytic and nowhere 0 in a neighbourhood of $(0, 0)$. \hfill $\square$

Corollary 3 ($k = 1$ in Weierstraß preparation, Implicit function theorem).

Let $f : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ and let $f$ be analytic in a neighbourhood of $(0, 0)$. Suppose

$$f(0, 0) = 0, \text{ but } \frac{d}{dy} f(0, 0) \neq 0.$$  

Then there exists a neighbourhood of 0 in $\mathbb{C}$ and a function $g(x)$ analytic on the neighbourhood with

1. $f(x, g(x)) = 0$, for all $x$ in the neighbourhood

2. if $f(x, y) = 0$ for $x, y$ sufficiently close to 0, then $y = g(x)$.

Proof. On the neighbourhood of $(0, 0)$, by Weierstraß preparation, we get

$$f(x, y) = (p_0(x) + y)r(x, y).$$

Now $r(x, y)$ is nowhere 0 on the neighbourhood, so $f(x, y) = 0$ if and only if $-p_0(x) = y$, so $g(x) = -p_0(x)$ will work. \hfill $\square$

Corollary 4 ($k = 2$ in Weierstraß preparation). Let $f : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ and let $f$ be analytic in a neighbourhood of $(0, 0)$. Suppose

$$f(0, 0) = \frac{d}{dy} f(0, 0) = 0, \text{ but } \frac{d^2}{dy^2} f(0, 0) \neq 0.$$  

Then in a neighbourhood of $(0, 0)$,

$$f(x, y) = (p_0(x) + p_1(x)y + y^2)r(x, y)$$

with $p_i$ analytic in neighbourhood and $r(x, y)$ nowhere 0 in the neighbourhood.
Now we can prove Proposition 1:

Proof of Proposition 1. As \( \exists \epsilon \) with \( E(\rho + \epsilon, T(\rho) + \epsilon) < \infty \) and we have nonnegative coefficients, we can choose a neighbourhood \( \mathcal{U} \) of \( (\rho, T(\rho)) \) such that \( E \) is analytic on \( \mathcal{U} \).

Let

\[
F(x, y) = y - E(x, y).
\]

Then \( F \) is analytic on \( \mathcal{U} \) and \( F(x, T(x)) = T(x) - E(x, T(x)) = 0 \) for \( |x| < \rho \). By Pringsheim’s Theorem, \( \rho \) is a singularity so the hypotheses of the implicit function theorem must be false at \( (\rho, T(\rho)) \); thus we must have \( \frac{d}{dy} F(\rho, T(\rho)) = 0 \).

\[
\frac{d}{dy} F(x, y) = 1 - \frac{d}{dy} E(x, y)
\]

We want to check that the hypotheses of Corollary 4 are satisfied:

\[
\frac{d^2}{dy^2} F(x, y) = -\frac{d^2}{dy^2} E(x, y) < 0
\]

for \( x, y > 0 \) (since we have nonnegative coefficients and at least one \( y^2 \) term), so in particular,

\[
\frac{d^2}{dy^2} F(\rho, T(\rho)) < 0
\]

thus

\[
F(x, y) = \left( p_0(x) + p_1(x)y + y^2 \right) r(x, y)
\]

with \( p_i \) analytic not 0 at \( \rho \) and \( r(x, y) \) analytic, nowhere 0 in a neighbourhood of \( (\rho, T(\rho)) \).

Let \( D(x) \) be the discriminant of \( P(x, y) \)

\[
D(x) = p_1(x)^2 - 4p_0(x)
\]

Next we want to check \( D(\rho) = 0, \frac{d}{dx} D(\rho) \neq 0 \). To see these, just calculate:

\[
F(x, T(x)) = 0, \text{ but } r(x, T(x)) \neq 0 \text{ for } x \text{ near } \rho, \text{ so}
\]

\[
p_0(\rho) + p_1(\rho)T(\rho) + T(\rho)^2 = 0. \tag{1}
\]

Also

\[
0 = \frac{d}{dy} F(\rho, T(\rho))
\]

\[
= \left( \frac{d}{dy} P(\rho, T(\rho)) \right) r(\rho, T(\rho)) + P(\rho, T(\rho)) \frac{d}{dy} r(\rho, T(\rho)) \\
\Rightarrow 0 = \frac{d}{dy} P(\rho, T(\rho)) = p_1(\rho) + 2T(\rho)
\]

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and subbing into (1) gives

\[ 0 = p_0(\rho) - \frac{p_1^2(\rho)}{2} + \frac{p_2^2(\rho)}{4} = p_0(\rho) - \frac{p_1^2(\rho)}{4} = -\frac{D(\rho)}{4} \]

so \( D(\rho) = 0 \). Now

\[
\frac{d}{dx}D(\rho) = 2p_1(\rho) \frac{d}{dx}p_1(\rho) - 4 \frac{d}{dx}p_0(\rho) = -4 \left( T(\rho) \frac{d}{dx}p_1(\rho) + \frac{d}{dx}p_0(\rho) \right)
\]

\[
\frac{d}{dx}F(\rho, T(\rho)) = -\frac{d}{dx}E(\rho, T(\rho)) < 0
\]

and

\[
\frac{d}{dx}F(\rho, T(\rho)) = \left( \frac{d}{dx}p_0(\rho) + T(\rho) \frac{d}{dx}p_1(\rho) \right) r(\rho, T(\rho)) + 0, \quad \text{since } P(\rho, T(\rho)) = 0,
\]

So

\[
\frac{d}{dx}D(\rho) = 4 \frac{d}{dx}E(\rho, T(\rho)) \neq 0.
\]

Thus \( D(\rho) = 0, \frac{d}{dx}D(\rho) \neq 0. \)

Returning to the previous calculation we know

\[ p_0(x) + p_1(x)T(x) + T(x)^2 = 0 \]

for \( x \) near \( \rho \), so

\[ T(x) = -\frac{p_1(x)}{2} + \frac{1}{2} \sqrt{D(x)}. \]

Since \( D(\rho) = 0 \) we can expand \( \sqrt{D(x)} \) around \( \rho \) to get

\[ D(x) = \sum_{k \geq 1} d_k(\rho - x)^k \]

and since \( \frac{d}{dx}D(\rho) \neq 0 \) we know \( d_1 \neq 0 \). So

\[
T(x) = -\frac{1}{2} p_1(x) + \left( \frac{1}{2} \sqrt{d_1} \sqrt{1 + \sum_{k \geq 1} \frac{d_{k+1}(\rho - x)^k}{d_1(\rho - x)^k}} \right) \sqrt{\rho - x}
\]

for \( x \) near \( \rho \).
2 Cauchy’s Theorems

Definition. Let \( \Omega \) be a connected open subset of \( \mathbb{C} \). A path is a function \( \gamma : [0, 1] \rightarrow \Omega \).

Definition. Two paths \( \gamma_1, \gamma_2 : [0, 1] \rightarrow \Omega \) with \( \gamma_1(0) = \gamma_2(0), \gamma_1(1) = \gamma_2(1) \) are homotopic (above right) if \( \exists h(x, y) \) continuous with image in \( \Omega \) such that

\[
\begin{align*}
  h(x, 0) &= \gamma_1(x) \\
  h(x, 1) &= \gamma_2(x) \\
  h(0, y) &= \gamma_1(0) \\
  h(1, y) &= \gamma_1(1).
\end{align*}
\]

Definition. A closed path has \( \gamma(0) = \gamma(1) \).

Definition. A simple path is 1-1 as a function.

Note. Being homotopic depends on \( \Omega \).

Definition. Integrals along paths are defined as you’d expect:

\[
\int_{\gamma} f(z)dz = \int_{0}^{1} f(\gamma(t))\gamma'(t)dt.
\]

Complex analysis is very rigid. Another important example of this is

Theorem 5. If \( f \) is analytic on \( \Omega \) and \( \gamma_1, \gamma_2 \) are homotopic in \( \Omega \) then

\[
\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.
\]
**Theorem 6** (Cauchy’s residue theorem). Let \( h(z) \) be meromorphic (i.e., holomorphic except possibly for finitely many poles) in \( \Omega \) and let \( \lambda \) be a positively oriented simple closed path in \( \Omega \). Let \( S \) be the set of poles of \( h \) inside the region enclosed by \( \lambda \). Then
\[
\frac{1}{2\pi i} \int_{\lambda} h(z) \, dz = \sum_{s \in S} \text{Res}_s h
\]
where \( \text{Res}_s h \) is the \([(z - s)^{-1}]\) in a Laurent expansion of \( h \) around \( s \).

**Proof.** (For just 1 pole at 0). So
\[
h(z) = \sum_{n=0}^{\infty} h_n z^n
\]
then
\[
\int_{\lambda} h(z) \, dz = \int_{\lambda} \sum_{\substack{n \geq -1 \\ n \neq -1}} h_n z^n \, dz + h_{-1} \int_{\lambda} \frac{dz}{z}
\]
and for \( n \neq -1, \)
\[
h_n \int_{\lambda} z^n \, dz = h_n \int_{0}^{1} e^{2\pi i nt} 2\pi i e^{2\pi it} \, dt, \quad \text{letting } \lambda(t) = e^{2\pi it}
\]
\[
= 2\pi i h_n \int_{0}^{1} e^{2\pi it(n+1)} \, dt
\]
\[
= 0,
\]
but
\[
\int_{\lambda} \frac{dz}{z} = \int_{0}^{1} e^{-2\pi it} 2\pi i e^{2\pi it} \, dt
\]
\[
= 2\pi i \cdot 1.
\]
So \( \int_{\lambda} h(z) \, dz = 2\pi i h_{-1}. \)

**Theorem 7** (Cauchy’s coefficient formula). Let \( f(z) \) be analytic in a region \( \Omega \) containing 0. Let \( \lambda \) be a positively oriented simple closed path in \( \Omega \). Then
\[
[z^n] f(z) = \frac{1}{2\pi i} \int_{\lambda} f(z) \frac{dz}{z^{n+1}}.
\]

**Proof.** Write
\[
f(z) = \sum_{\ell=0}^{\infty} f_\ell z^\ell
\]
then
\[
\frac{f(z)}{z^{n+1}} = \sum_{\ell=-n}^{\infty} f_{\ell+n+1} z^\ell
\]
and so the residue is \( f_n \), so the result is an application of Cauchy’s residue theorem.
3 Transfer Theorems

Now we can use this to get a nice transfer theorem.

**Definition.** A *delta neighbourhood of* \( \rho \) is a region as illustrated

![Diagram of a delta neighbourhood](image)

**Note.** Stirling's formula (with the constant) says for \( \alpha \in \mathbb{R} \setminus \mathbb{Z} \)

\[
[x^n](\rho - x)^\alpha \sim \frac{\rho^\alpha}{\Gamma(-\alpha)} \rho^{-n} n^{-\alpha-1}.
\]

**Theorem 8** (Transfer theorem of Flajolet and Odlyzko). Let \( 0 < \rho < \infty \) and suppose \( f \) is analytic on \( \Delta - \rho \) with \( \Delta \) a delta neighbourhood of \( \rho \) and \( f(x) \sim K(\rho - x)^\alpha \) as \( x \to \rho \) in \( \Delta \) with \( \alpha \in \mathbb{R} \setminus \mathbb{Z} \), then

\[
[x^n]f(x) \sim [x^n] K(\rho - x)^\alpha \\
\sim \frac{K \rho^\alpha}{\Gamma(-\alpha)} \rho^{-n} n^{-\alpha-1}
\]

**Sketch of proof.** Use the following contour:

![Sketch of proof diagram](image)
Write

\[
\gamma = \begin{cases} 
\gamma_1 = \{x : |x - \rho| = \frac{1}{n}, |\arg(x - \rho)| \geq \theta\} & \text{inner circle} \\
\gamma_2 = \{x : \frac{1}{n} \leq |x - \rho|, |x| \leq \rho + \eta, \arg(x - \rho) = \theta\} & \text{straight piece} \\
\gamma_3 = \{x : |x| = \rho + \eta, |\arg(x - \rho)| \geq \theta\} & \text{outer circle} \\
\gamma_4 = \{x : \frac{1}{n} \leq |x - \rho|, |x| \leq \rho + \eta, \arg(x - \rho) = -\theta\} & \text{straight piece}
\end{cases}
\]

Wlog scale so \(\rho = 1\). Now bound each piece:

- \(\gamma_1\) bound by (length of path)(max of integrand)
- \(\gamma_2, \gamma_4\) are tricky ones, like \(\Gamma\)-function integral
- \(\gamma_3\) easy, as \(f\) bounded so only care about \(\int \frac{1}{z^n} \)

\[
\blacksquare
\]

Note. If there are \(> 1\) singularities on the circle of convergence, but only finitely many, we can give the same argument using the following contour (simply add more keyholes) to get the same result:

\[
\text{(Image of contour with additional keyholes)}
\]

References
