Math 821 Combinatorics Notes

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1 Hopf Algebras and the Antipode

Another thing which comes for free in the case of graded connected bi-algebras is that they are Hopf algebras

Definition Let A, B be k-vector spaces. Then $Hom(A, B)$ is the space of k-linear maps from A to B

Proposition 1.1. *Let C be a k-coalgebra and A a k-algebra. Then $Hom(C, A)$ is itself an algebra, called the convolution algebra, with convolution product*

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[rrr, bend right, "\text{convolution } f * g"'] & C \otimes C \arrow[r, "f \otimes g"] & A \otimes A \arrow[r, "\cdot"] & A
\end{tikzcd}$$

the identity of convolution product is $u \cdot \epsilon$

Corollary 1.2. *If A is a bialgebra then $Hom(A, A)$ has a convolution algebra structure*

Definition A bialgebra A is a Hopf algebra if there exists an $S \in Hom(A, A)$ such that S is a two-sided inverse of id_A in the convolution algebra.

$$\begin{tikzcd}
& A \otimes A \arrow[rr, "S \otimes id_A"] & & A \otimes A \arrow[dr, "\cdot"] & \\
A \arrow[ur, "\Delta"] \arrow[rr, "\epsilon"] \arrow[dr, "\Delta"'] & & \text{k} \arrow[rr, "u"] & & A \\
& A \otimes A \arrow[rr, "id_A \otimes S"'] & & A \otimes A \arrow[ur, "\cdot"'] &
\end{tikzcd}$$

We call S the antipode

Proposition 1.3. *The antipode S of a Hopf algebra A is an algebra anti-automorphism. That is, $S(\mathbb{I}) = \mathbb{I}$ and $S(ab) = S(b)S(a)$*

Proof. First we take $\mathbb{I}$ through the commutative diagram.

$$\begin{tikzcd}
& \mathbb{I} \otimes \mathbb{I} \arrow[rr, "S \otimes id_A"] & & S(\mathbb{I}) \otimes \mathbb{I} \arrow[dr, "\cdot"] & \\
\mathbb{I} \arrow[ur, "\Delta"] \arrow[rr, "\epsilon"] & & 1 \arrow[rr, "u"] & & \mathbb{I} = S(\mathbb{I})
\end{tikzcd}$$

Now consider $Hom(A \otimes A, A)$. This is a co-algebra so it has a convolution product. Write it as $\odot$ to keep it distinct from $*$. Define:

$$\begin{aligned} f :& A \otimes A \to A & g :& A \otimes A \to A & h :& A \otimes A \to A \\ & a \otimes b \to ab & & a \otimes b \to S(b)S(a) & & a \otimes b \to S(ab) \end{aligned}$$

We claim $h \odot f = u_A \epsilon_{A \otimes A} = f \odot g$. To prove the claim we now calculate. Let

$$\begin{aligned} \Delta(a) &= \sum_i a_{i,1} \otimes a_{i,2} \quad , \quad \Delta(b) = \sum_j b_{j,1} \otimes b_{j,2} \\ \Rightarrow \Delta(ab) &= \sum_{i,j} a_{i,1} b_{j,1} \otimes a_{i,2} b_{j,2} \end{aligned}$$

So

$$\begin{aligned} u_A \epsilon A \otimes A(a \otimes b) &= u_A(\epsilon_A(a)\epsilon_A(b)) \\ &= u_A(\epsilon_A(ab)) \end{aligned}$$

Also

$$\begin{aligned} (h \odot f)(a \otimes b) &= \cdot_A (h \otimes f) \left(\sum_{i,j} a_{i,1} b_{j,1} \otimes a_{i,2} b_{j,2} \right) \\ &= \sum_{i,j} h(a_{i,1} \otimes b_{j,1}) f(a_{i,2} b_{j,2}) \\ &= \sum_{i,j} S(a_{i,1} b_{j,1}) a_{i,2} b_{j,2} = (S * id)(ab) \end{aligned}$$

Similarly, we get that

$$
\begin{aligned}
(f \odot g)(a \otimes b) &= \cdot_A (f \otimes g) \left(\sum_{i,j} a_{i,1} b_{j,1} \otimes a_{i,2} b_{j,2} \right) \\
&= \sum_{i,j} f(a_{i,1} \otimes b_{j,1}) g(a_{i,2} b_{j,2}) = \sum_{i,j} a_{i,1} \left[b_{j,1} S(b_{j,2}) \right] S(a_{i,2}) \\
&= \sum_i a_{i,1} \underbrace{\left(\sum_j b_{j,1} S(b_{j,2}) \right)}_{(id * S)(b) = u_A \epsilon_A(b)} S(a_{i,2}) \\
&= u_A \epsilon_A(b) \sum_i a_{i,1} S(a_{i,2}) = u_A \epsilon_A(b) \sum_i a_{i,1} S(a_{i,2}) (id * S)(a) = u_A(\epsilon(ab))
\end{aligned}
$$

□

This has a lot of useful consequences.

Corollary 1.4. *If A is commutative then $S^2 = S \circ S = id_A$*

Proof.

$$
\begin{aligned}
S * S^2(a) &= \sum_i S(a_{i,1}) S^2(a, 2) \\
&= S \left(\sum_i S(a_{i,2}) a_{i,1} \right) && \text{by proposition} \\
&= S \left(\sum_i a_{i,2} S(a_{i,2}) \right) && \text{by commutativity} \\
&= S(id * S)(a) = S(u(\epsilon(a))) = u(q(a)) && S(\mathbb{I}) = \mathbb{I}
\end{aligned}
$$

So $S * S^2 = u\epsilon = id * S$. Therefore $id = id * (u\epsilon) = (id * S) * S^2 = u\epsilon * S^2 = S^2$ □

Corollary 1.5. *If A is commutative then $S^2 = id_A$.*

Proof.

$$
\begin{aligned}
(S * S^2)(a) &= \sum_i S(a_{i,1}) S^2(a_{i,2}) \\
&= \sum_i S^2(a_{i,2}) S(a_{i,1}) && \text{by commutativity} \\
&= S \left(\sum_i S(a_{i,2}) a_{i,1} \right) \\
&= S(S * id)(a) = S(u\epsilon(a)) = u(\epsilon(a)) && \text{as } S(\mathbb{I}) = \mathbb{I}
\end{aligned}
$$

□

Corollary 1.6. *If A is a graded connected bi-algebra than A has a unique antipode S. Furthermore S is a graded map, so A is a graded Hopf algebra*

Proof. Converting $S * id = u\epsilon$ into a recurrence. Write

$$A = \bigoplus_{n=0}^{\infty} A_n$$

Base case: $n = 0$. $A_0 = k$ by proposition $S(\mathbb{I}) = \mathbb{I})$ so $S|_{A_0} = id$
For $x_i \in \bigoplus_{i=1}^{\infty} A_i$ write $\Delta(x) = \mathbb{I} \otimes x + x \otimes 1 + \tilde{\Delta}(x)$ and write

$$\tilde{\Delta}(x) = \sum_i x_{i,1} \otimes x_{i,2}$$

$$0 = u\epsilon(x) = (S * id)(x) = x + S(x) + \sum_i S(x_{i,1})x_{i,2}$$

so

$$S(x) = -x - \sum_i S(x_{i,1})x_{i,2}$$

and if x is homogeneous of degree n then the $x_{i,1}$ and $x_{i,2}$ are degree less than n. This determines S recursively. □

1.1 Examples

Example words = TV

$$S(\mathbb{I}) = \mathbb{I}$$

And for a word of one letter

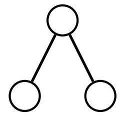
$$\begin{aligned} S(x) = -x, \Delta(x) &= \mathbb{I} \otimes x + x \otimes \mathbb{I} \\ \implies S(x_1 \dots x_k) = S(x_k) \dots S(x_1) &= (-1)^k x_k \dots x_1 \end{aligned}$$

antipode reverses the word and puts a sign on it.

Example Connes-Kreimer Hopf algebra of rooted trees.
For a tree t

$$S(t) = -t - \sum_{\substack{\text{admissible cuts } i \\ c \neq \emptyset \\ c \neq root}} S(P_c(t))R_c(t)$$

Sub-example
For notational convenience we denote

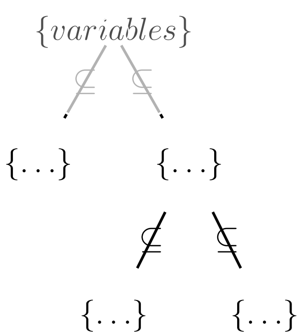
$k_1 =$  $k_2 =$  $P_2 =$ 

$$
\begin{aligned}
S(k_1) &= -k_1 \\
\\
S(k_2) &= -k_2 - S(k_1)k_1 = -k_2 + k_1k_1 \\
\Delta(k_2) &= k_2 \otimes \mathbb{I} + \mathbb{I} \otimes k_2 + k_1 \otimes k_1 \\
\\
\Delta(P_2) &= P_2 \otimes \mathbb{I} + \mathbb{I} \otimes P_2 + 2\,(k_1 \otimes k_2) + k_1 \times k_1 \otimes k_1 \\
S(P_2) &= -P_2 - S(k_1)k_2 - S(k_1)^2 k_1 \\
&= -P_2 + 2k_1k_2 - k_1k_1k_1
\end{aligned}
$$

What does S actually do? In pertubative quantum field theory there are formal integrals indexed by Feynman graphs. The ones you care about diverge.

$$\int (\mu - \mu \cdot 1)$$

You can fix these. If the only problem occurs when variations get large then you can fix it using traditional means. But sometimes the integral also diverges when some subset of variables gets large. We can describe these with a tree structure.



We need to subtract off stuff to fix the sub-divergences. The antipode says exactly how to do this

$$S(t) = -t \;\; - \sum_{\substack{\text{admissible cuts } i \\ c \neq \emptyset \\ c \neq root}} \underbrace{S(P_c(t))}_{\substack{\text{subtracting off for} \\ \text{this subdivergence}}} \cdot \underbrace{R_c(t)}_{\text{leave this part alone}}$$

2 References

For everything but the last part: *Reiner up to 1.5*