MATH 821, SPRING 2012, ASSIGNMENT 1 SOLUTIONS

(1) First

$$A(x) = \sum_{n\geq 0} Z(D_n, B(x), B(x)^2, \dots, B(x)^n)$$

So let's focus on the cycle index polynomials.

$$\begin{aligned}
\sum_{n\geq 0} Z(D_n, s_1, \dots, s_n) &= \frac{1}{|D_n|}\sum_{\sigma\in D_n} s_1^{j_1(\sigma)}\cdots s_n^{j_n(\sigma)} \\
&= \frac{1}{|D_n|}\sum_{\sigma\in C_n} s_1^{j_1(\sigma)}\cdots s_n^{j_n(\sigma)} + \frac{1}{|D_n|}\sum_{\sigma\in C_n} s_1^{j_1(\tau\sigma)}\cdots s_n^{j_n(\tau\sigma)} \\
&= \frac{1}{2}Z(C_n, s_1, \dots, s_n) + \frac{1}{|D_n|}\sum_{\sigma\in C_n} s_1^{j_1(\tau\sigma)}\cdots s_n^{j_n(\tau\sigma)}
\end{aligned}$$

where τ is the order two generataor of the dihedral group (the flip generator).

If n is even, then we have two kinds of elements in τC_n, flips across the axis through opposite edges of the cycle and flips across the axis through opposite vertices of the cycle. The first of these consists of $n/2$ transpositions in disjoint cycle representation, while the second of these consists of $(n-2)/2$ transpositions and 2 fixed points. Exactly half of the elements of τC_n are of each kind. Therefore for n even

$$\frac{1}{|D_n|}\sum_{\sigma\in C_n} s_1^{j_1(\tau\sigma)}\cdots s_n^{j_n(\tau\sigma)} = \frac{1}{4}\left(s_1^2 s_2^{(n-2)/2} + s_2^{n/2}\right)$$

If n is odd, then all elements in τC_n consist of a flip through an axis which goes through one edge and an opposite vertex of the cycle. Such a flip has $(n-1)/2$ transpositions and one fixed point in its disjoint cycle representation. Therefore for n odd

$$\frac{1}{|D_n|}\sum_{\sigma\in C_n} s_1^{j_1(\tau\sigma)}\cdots s_n^{j_n(\tau\sigma)} = \frac{1}{2}\left(s_1 s_2^{(n-1)/2}\right)$$

The result follows.

(2) (a) $\mathcal{B} = \mathcal{Z} + \mathcal{Z}\times\mathcal{B}^2$ (you could also write one allowing the empty tree, but watch out you don't accidentally allow empty children).

(b) We have $B(x) = x + xB(x)^2$. Using the quadratic formula we get

$$B(x) = \frac{1\pm\sqrt{1-4x^2}}{2x}$$

We must want the negative sign since if we sub in $x = 0$ we should get 0, which the negative sign gives, while the positive sign gives a pole at 0. So

$$B(x) = \frac{1-\sqrt{1-4x^2}}{2x}$$

So $[x^0]B(x) = 0$ and for $n > 0$

$$\begin{aligned}
[x^n]B(x) &= -\frac{1}{2}[x^{n+1}](1-4x^2)^{1/2} \\
&= \begin{cases} -\frac{1}{2}\binom{1/2}{(n+1)/2}(-4)^{(n+1)/2} & \text{if } n+1 \text{ is ev en} \\ 0 & \text{if } n+1 \text{ is odd} \end{cases} \\
&= \begin{cases} -\frac{1}{2}\binom{1/2}{m}(-4)^m & \text{if } n = 2m-1 \\ 0 & \text{if } n = 2m \end{cases} \\
&= \begin{cases} -\frac{1}{2}\left(\frac{-1}{4}\right)^{m-1}\frac{1}{2m}\binom{2m-2}{m-1}(-4)^m & \text{if } n = 2m-1 \\ 0 & \text{if } n = 2m \end{cases} \\
&= \begin{cases} \frac{1}{m}\binom{2m-2}{m-1} & \text{if } n = 2m-1 \\ 0 & \text{if } n = 2m \end{cases}
\end{aligned}$$

(c) The trees of this problem only exist when n is odd, while the trees from lecture exist for all $n > 0$. In both cases, however, they are counted by Catalan numbers. Specifically, if $\mathcal{T}$ is the class of binary rooted trees from lecture, then for $n > 0$,

$$[x^n]T(x) = \frac{1}{n+1}\binom{2n}{n} = C_n = [x^{2n+1}]B(x)$$

To be clear the trees from this problem are the class $\mathcal{B}$ and the ones from lecture (with distinct left and right children) are $\mathcal{T}$.
To find the bijection first note that a tree $t \in \mathcal{B}_{2n+1}$ has $n+1$ leaves. We can prove this by induction. It is true for $n = 0$ since the only tree in that case is the one vertex tree which has one leaf. Take $n > 0$ and suppose it is true for $k < 2n+1$. Since $n > 0$ this tree has two nonempty children of its root, say of sizes $2k+1$ and $2\ell+1$ where $2(k+\ell)+2+1 = 2n+1$ so $k+\ell+1 = n$. By induction the subtrees at the root have $k+1$ and $\ell+1$ leaves respectively, so the tree has $k+1+\ell+1 = n+1$ leaves.
Consider the map

$$f : \mathcal{B} \to \mathcal{T}$$

where $f(t)$ is the tree obtained from t by removing all the leaves of t. This gives a tree in $\mathcal{T}$ since t originally had distinct left and right children, and upon removing leaves, some of them now may be empty. By the observation of the previous paragraph

$$f : \mathcal{B}_{2n+1} \to \mathcal{T}_n$$

The inverse map

$$g : \mathcal{T} \to \mathcal{B}$$

is defined as follows. Let $g(t)$ be the tree obtained from t by putting a new leaf wherever a vertex of t has an empty child (including the original leaves of t which now receive two new leaves as children). The resulting tree has every vertex originally from t having degree 2 and the new vertices have degree 0, so this is a tree in $\mathcal{B}$.

By construction the maps are inverses of each other, so so give the desired bijection.

(3) (a) $\mathcal{C} = \textsc{Seq}(\textsc{Seq}_{\geq 1}(\mathcal{Z}))$. Thus

$$C(x) = \frac{1}{1 - \frac{x}{1-x}} = \frac{1-x}{1-2x}$$

(b) $\mathcal{C}_{\substack{\text{at most } k \text{ parts} \\ \text{odd parts}}} = \textsc{Seq}_{\leq k}(\textsc{Seq}_{\text{odd}}(\mathcal{Z})) = \sum_{i=0}^{k} \left(\mathcal{Z} \times \textsc{Seq}(\mathcal{Z}^2)\right)^i$. Thus

$$C(x) = \sum_{i=0}^{k} \left(\frac{x}{1-x^2}\right)^i$$

(4) (a) An element of $\Theta(\mathcal{C})_n$ is a pair (C, z) with $C \in \mathcal{C}_n$ and z an atom of C. For each $C \in \mathcal{C}_n$ there are n atoms making it up, and so $|\Theta(\mathcal{C})_n| = n|\mathcal{C}_n|$. Therefore Θ is admissible

(b) From the observation of the previous part

$$A(x) = \sum_{n=0}^{\infty} n a_n x^n = x \frac{d}{dx} C(x)$$

(5) *solutions will vary*