MATH 821, SPRING 2012, ASSIGNMENT 2 SOLUTIONS

(1) (a) Let $\mathcal{C}$ be an unlabelled combinatorial class with $\mathcal{C}_0 = \emptyset$ and let $\mathcal{B} = \textsc{MSet}(\mathcal{C})$. Then

$$B(x) = \prod_{c \in \mathcal{C}} (1 - x^{|x|})^{-1}$$

since each element can appear any number of times. Therefore

$$\begin{aligned} B(x) &= \prod_{c \in \mathcal{C}} (1 - x^{|x|})^{-1} \\ &= \prod_{n=1}^{\infty} (1 - x^n)^{-c_n} \\ &= \exp\left(\log\left(\prod_{n=1}^{\infty} (1 - x^n)^{-c_n}\right)\right) \\ &= \exp\left(\sum_{n=1}^{\infty} c_n \log(1 - x^n)^{-1}\right) \\ &= \exp\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{c_n z^{nk}}{k}\right) \\ &= \exp\left(\sum_{k=1}^{\infty} \frac{C(x^k)}{k}\right) \end{aligned}$$

The other one is harder. Let W be a set of words of length n. Let S_n be the symmetric group on $\{1, \dots, n\}$ which we can view acting on W. Let $d(W)$ be the number of elements of W which are fixed only by the identity element of S_n. Let $s(\sigma)$ be the of $\sigma \in S_n$. Then I claim

$$d(W) = \sum_{w \in W} \sum_{\sigma \in \mathrm{Stab}_{S_n}(w)} (-1)^{s(\sigma)}$$

Proof of claim: If w is fixed only by the identity then the sum gives just $(-1)^0 = 1$. Suppose w is fixed by some other permutation, then it must have some identical letters, and our goal is to prove that the sum gives 0. To set notation, say a appears $k > 1$ times in w. Let v be w with all instances of a removed.

Then

$$\sum_{\sigma\in \mathrm{Stab}_{S_n}(w)} (-1)^{s(\sigma)} = \left(\sum_{\sigma_1\in \mathrm{Stab}_{S_{n-k}}(v)} (-1)^{s(\sigma_1)}\right)\left(\sum_{\sigma_2\in \mathrm{Stab}_{S_k}(a^k)} (-1)^{s(\sigma_2)}\right)$$
$$= \left(\sum_{\sigma_1\in \mathrm{Stab}_{S_{n-k}}(v)} (-1)^{s(\sigma_1)}\right)\left(\sum_{\sigma_2\in S_k} (-1)^{s(\sigma_2)}\right)$$

where a^k is the word consisting of k copies of a. Next notice that since $k \geq 2$, then half of the elements of S_k have even sign and half have odd sign, so the second factor above is 0.
This completes the proof of the claim.
You might wonder where this claim came from; the answer is that I just worked my way backwards through the proof of the Polya enumeration theorem until I ended up with this statement.

We need to observe one final thing about permutations before we return to combinatorial classes. We need to know that the sign of $\sigma \in S_n$ is $(-1)^{n+c(\sigma)}$ where $c(\sigma)$ is the number of cycles of σ. To see this note that $n + c(\sigma)$ counts each cycle with one more than its length, and thus $n+c(\sigma)$ is odd iff and onyl if the permutation has an odd number of even cycles, which is one of the standard formulations of oddness of a permutation.

Let $\mathcal{C}$ be a combinatorial class with $\mathcal{C}_0 = \emptyset$, and let $\mathcal{B} = \mathrm{PS{\scriptsize ET}}_n(\mathcal{C})$. As in the proof of the Polya enumeration theorem let $b_k(\alpha)$ be the number of elements of $\mathcal{B}$ fixed by $\alpha \in S_n$. The above observations applied to this case give

$$b_k = \frac{1}{n!}\sum_{\sigma\in S_n} b_k(\sigma)$$

since every set of n elements can be ordered in $n!$ ways to give a word with distinct letters. Now continue in the proof of the Pólya enumeration theorem using this fact in place of Burnside's lemma. We get

$$B(x) = \sum_{k=0}^{\infty}\frac{1}{n!}\sum_{\sigma\in S_n}(-1)^{n+c(\sigma)} b_k(\sigma)x^k$$
$$= \frac{1}{n!}\sum_{\sigma\in S_n}(-1)^{n+c(\sigma)}\sum_{k=0}^{\infty} b_k(\sigma)x^k$$

As in class, $b \in \mathcal{B}$ fixed by σ must be constant on each cycle of σ and so

$$\begin{aligned}
(-1)^{n+c(\sigma)} \sum_{k=0}^{\infty} b_k(\sigma) x^k &= (-1)^{n+c(\sigma)} \prod_{k=1}^{n} \left(\sum_{i=1}^{\infty} c_i x^{ki} \right)^{j_k(\sigma)} \\
&= (-1)^{\sum_{k=1}^{n} (k+1) j_k(\sigma)} \prod_{k=1}^{n} \left(C(x^k) \right)^{j_k(\sigma)} \\
&= \prod_{k=1}^{n} \left((-1)^{k+1} C(x^k) \right)^{j_k(\sigma)}
\end{aligned}$$

Therefore

$$B(x) = Z(S_n; C(x), -C(x^2), \ldots, (-1)^{n+1} C(x^n))$$

Now redefinding $\mathcal{B} = \text{PSet}(\mathcal{C})$ summing we have

$$B(x) = \sum_{n=0}^{\infty} Z(S_n; C(x), -C(x^2), \ldots, (-1)^{n+1} C(x^n))$$

Then calculating exactly as in the MSet case we get

$$B(x) = \exp\left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1} B(x^k)}{k} \right)$$

(b) The signs are acting as a kind of inclusion-exclusion. In fact the claim in the proof of the previous part can be viewed as an inclusion-exclusion tailored to the symmetric group.

(2) (a) See the proof of Lemma 39 in Jason Bell, Stanley Burris, and Karen Yeats, Counting Rooted Trees: The Universal Law $t(n) \sim C\rho^n n^{3/2}$. Elec. J. Combin. 13 (2006), #R63. (Also arXiv:math.CO/0512432.) Note that the hypotheses of Lemma 39 have nonnegative coefficients, but that this is unnecessary for the proof.

(b) See the proof of Lemma 57 in the same paper.

(3) (a) From $\mathcal{T} = \mathcal{E} + \mathcal{Z}\mathcal{T}^2$ we can't easily spot the leaves, so rewite this as follows

$$\begin{aligned}
\mathcal{T} &= \mathcal{E} + \mathcal{U} \\
\mathcal{U} &= \mathcal{Z} + 2\mathcal{Z} \times \mathcal{U} + \mathcal{Z} \times \mathcal{U}^2
\end{aligned}$$

Of the occurences of $\mathcal{Z}$, the only one which is a leaf is the one in the first term of the expression for $\mathcal{U}$. Therefore

$$T(x,y) = 1 + xy + 2x(T(x,y) - 1) + x(T(x,y) - 1)^2$$

Expanding out this is

$$\begin{aligned}
T(x,y) &= 1 + xy - 2x + x + 2xT(x,y) - 2xT(x,y) + xT(x,y)^2 \\
&= 1 + x(y-1) + xT(x,y)^2
\end{aligned}$$

(b) First solve the equation in the previous part for $T(x,y)$,

$$T(x,y) = \frac{1 - \sqrt{1 - 4x(1 + x(y-1))}}{2x}$$

where we must have the minus sign as for $y = 1$ we must agree with the calculation from the beginning of class. The number of trees of size n is (as calculated in class)

$$[x^n]T(x,1) = \frac{1}{n+1}\binom{2n}{n}$$

By taking a y derivative of $T(x,y)$ we weight each tree by its number of leaves. Thus the total number of leaves among all trees of size n is

$$\begin{aligned}
[x^n]\frac{d}{dy}T(x,1) &= [x^n]\frac{-1}{4x}(1 - 4x(1 + x(y-1)))^{-1/2}(-4x^2)|_{y=1} \\
&= [x^n]x(1-4x)^{-1/2} \\
&= [x^{n-1}](1-4x)^{-1/2} \\
&= (-4)^{n-1}\binom{-1/2}{n-1} \\
&= \binom{2n-2}{n-1}
\end{aligned}$$

Thus the average number of leaves in a tree of size n is

$$\frac{\binom{2n-2}{n-1}}{\frac{1}{n+1}\binom{2n}{n}} = \frac{(2n-2)!n!n!(n+1)}{(n-1)!(n-1)!(2n)!} = \frac{n^2(n+1)}{2n(2n-1)} = \frac{n(n+1)}{2(2n-1)}$$

(4) (a) As a labelled class

$$\mathcal{D}^{(r)} = \textsc{Set}(\textsc{DCyc}_{>r}(\mathcal{Z}))$$

(b) So $\mathcal{D}^{(1)} = \textsc{Set}(\textsc{DCyc}_{>1}(\mathcal{Z}))$ giving

$$\begin{aligned}
D^{(1)}(x) &= \exp\left(\sum_{i>1}\frac{x^i}{i}\right) \\
&= \exp\left(\log\left(\frac{1}{1-x}\right) - x\right) \\
&= \frac{1}{1-x}\exp(-x)
\end{aligned}$$

so

$$\begin{aligned}\frac{d_n^{(1)}}{n!} &= [x^n]\frac{1}{1-x}\exp(-x)\\ &= [x^n](1+x+x^2+\cdots+x^n)\exp(-x)\\ &= \sum_{i=0}^{n}[x^i]\exp(-x)\\ &= \sum_{i=0}^{n}\frac{(-1)^i}{i!}\end{aligned}$$

This is the truncation of the series of $\exp(-1)$. Therefore

$$\lim_{n\to\infty}\frac{d_n^{(1)}}{n!} = \sum_{i=0}^{\infty}\frac{(-1)^i}{i!} = e^{-1}$$

while

$$\lim_{n\to\infty} d_n^{(1)} = \infty$$

(5) (a) We could build the class $\mathcal{I}$ of positive integers as

$$\mathcal{I} = \text{MSet}_{\geq 1}(\mathcal{Z})$$

(MSet and Seq give the same counts when their argument is just an atom). But a multiset of identical elements has only one labelling up to isomorphism. So this class has one element of each positive size both in the labelled and the unlabelled case.

(b) If we instead build the positive integers as

$$\mathcal{I}' = \text{Seq}_{\geq 1}(\mathcal{Z})$$

Then labelling means labelling each element of a sequence, and every sequence of length n has $n!$ distinct labellings since all permutations are distinct. In fact this is one way to view permutations as a labelled combinatorial class. So this class has one element of each positive size in the unlabelled case and $n!$ elements of size n in the labelled case.

(c) Let's take non-plane rooted trees with no empty tree. Then in the unlabelled case we have

$$\mathcal{T}_u = \mathcal{Z} \times \text{MSet}(\mathcal{T}_u)$$

while in the labelled case we have

$$\mathcal{T}_\ell = \mathcal{Z} \times \text{Set}(\mathcal{T}_\ell)$$

So

$$\begin{aligned}T_u(x) &= x\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}T_u(x^n)\right)\\ T_\ell(x) &= x\exp(T_\ell(x))\end{aligned}$$

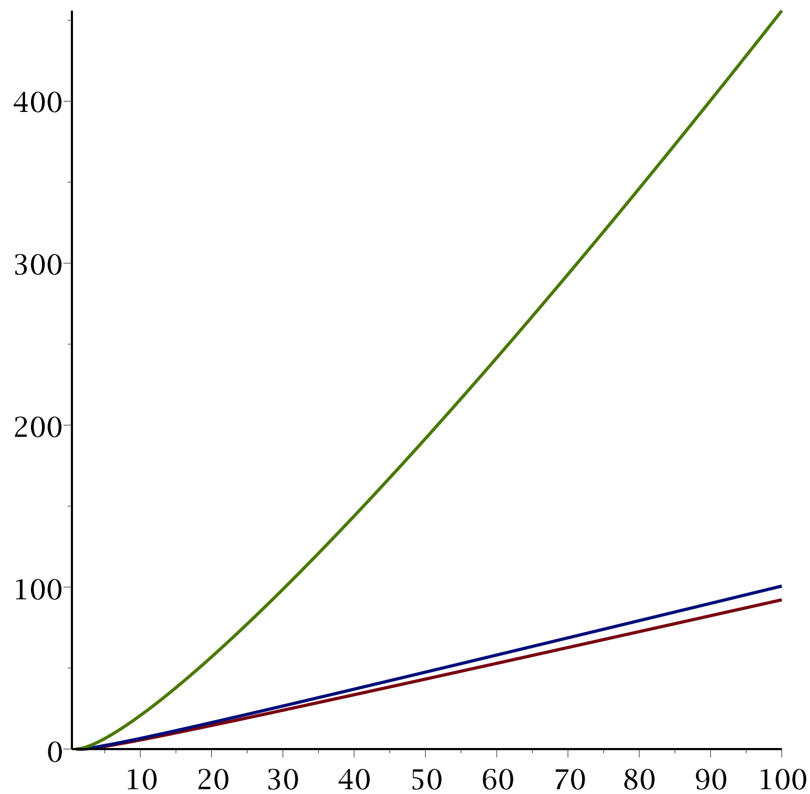
Using the following Maple commands I can get the two generating functions expanded out to 100 terms

```
with(combstruct);
Tspec := {T = Prod(Z,Set(T))};
Order := 101;
Tu := gfseries(Tspec, unlabelled, x)[T(x)];
Tl := gfseries(Tspec, labelled, x)[T(x)];
```

Now we need to fuss with them to get it ready to plot. Here's one way

```
Au := [];
Alwithfac := [];
Alnofac := [];
for i from 1 to 100 do
Au := [op(Au), [i, evalf(log(coeff(Tu, x, i)))]];
Alwithfac := [op(Alwithfac), [i,evalf(log(coeff(Tl, x, i)))]];
Alnofac := [op(Alnofac), [i,evalf(log(i!*coeff(Tl, x, i)))]];
end;
plot({Au, Alwithfac, Alnofac});
```

which gives the plot on the final page. The green line in the plot is the log of the counts for the labelled class, the blue line is the log of the counts of the unlabelled class, and the red line is the log of the coefficients of the exponential generating function for the labelled class. The blue line is strictly between the other two, so this class satisfies the requirement of this question.



400
300
200
100
0
10 20 30 40 50 60 70 80 90 100