MATH 821, SPRING 2012, ASSIGNMENT 3 SOLUTIONS

(1) First we have a trilinear map

$$f : V_1 \times V_2 \times V_3 \to V_1 \otimes (V_2 \otimes V_3)$$

given by $f(v_1, v_2, v_3) = v_1(\otimes v_2 \otimes v_3)$. Fixing any value v_3 in the third component $f(\cdot, \cdot, v_3)$ is bilinear and so by the universal property of tensor products we have a linear map

$$\phi_{v_3} : V_1 \otimes V_2 \to V_1 \otimes (V_2 \otimes V_3)$$

given by $\phi_{v_3}(v_1 \otimes v_2) = f(v_1, v_2, v_3) = v_1 \otimes (v_2 \otimes v_3)$.

Now consider

$$g : (V_1 \otimes V_2) \times V_3 \to V_1 \otimes (V_2 \otimes V_3)$$

given by $g(v, v_3) = \phi_{v_3}(v)$ for $v \in V_1 \otimes V_2$. Since f was linear in the third coordinate, g is linear in the third coordinate when applied to pure tensors and hence always. g is linear in the first coordinate since ϕ_{v_3} is linear for each v_3. Thus g is bilinear and so by the universal property of tensor products we have a linear map

$$\psi : (V_1 \otimes V_2) \otimes V_3 \to V_1 \otimes (V_2 \otimes V_3)$$

given by $\psi((v_1 \otimes v_2) \otimes v_3) = v_1 \otimes (v_2 \otimes v_3) = f(v_1, v_2, v_3)$. The same argument with the parenthesization the other way gives the inverse map and hence this is an isomorphism proving the result.

(2) Let $f, g, h \in \mathrm{Hom}(A, A)$. Then

$$\begin{aligned}
(f \star g) \star h &= \cdot((f \star g) \otimes h)\Delta \\
&= \cdot(\cdot \otimes \mathrm{Id})(f \otimes g \otimes h)(\Delta \otimes \mathrm{Id})\Delta \\
&= \cdot(\mathrm{Id} \otimes \cdot)(f \otimes g \otimes h)(\mathrm{Id} \otimes \Delta)\Delta \\
&= f \star (g \star h)
\end{aligned}$$

by associativity of $\cdot$ and coassociativity of Δ. Also

$$\begin{aligned}
f \star (u \circ \epsilon) &= \cdot(f \otimes (u \circ \epsilon))\Delta \\
&= \cdot(\mathrm{Id} \otimes u)(f \otimes \mathrm{Id})(\mathrm{Id} \otimes \epsilon)\Delta \\
&= f
\end{aligned}$$

by the unit and counit properties, and similarly on the other side

$$\begin{aligned}
(u \circ \epsilon) \star f &= \cdot((u \circ \epsilon) \otimes f)\Delta \\
&= \cdot(u \otimes \mathrm{Id})(\mathrm{Id} \otimes f)(\epsilon \otimes \mathrm{Id})\Delta \\
&= f
\end{aligned}$$

So $\mathrm{Hom}(A, A)$ is an algebra under the convolution product.

(3) There are a number of ways to approach this. Here's the very explicit approach, probably not the prettiest, but you just do it:

Let S° be the antipode of H°, and similarly for the other structure functions.

Let $\{a_i\}$ be a basis of H with $a_0 = 1$ and $\{f_i\}$ the dual basis of H°. Let $\Delta(a_i) = \sum_{j,k} c^{(i)}_{j,k} a_j \otimes a_k$ and $a_j \cdot a_k = \sum_i d^{(i)}_{j,k} a_i$. We know from class that $\Delta^\circ(f_i) = \sum_{j,k} d^{(i)}_{j,k} f_j \otimes f_k$ and $f_j \cdot^{\otimes} f_k = \sum_i c^{(i)}_{j,k} f_i$.

Write $S^\circ(a_i) = \sum_\ell e^{(i)}_\ell a_\ell$. Then

$$\cdot(S \otimes \mathrm{Id})\Delta = u \circ \epsilon$$

so for $i > 0$

$$0 = \cdot(S \otimes \mathrm{Id})\Delta(a_i) = \sum_{j,k,\ell,m} d^{(m)}_{\ell,k} e^{(j)}_\ell c^{(i)}_{j,k} a_m$$

so since the a_i form a basis

$$\sum_{j,k,\ell,m} d^{(m)}_{\ell,k} e^{(j)}_\ell c^{(i)}_{j,k} = 0 \tag{1}$$

I claim that $S^\circ = S^*$ (where $S^*(f)(a) = f(S(a))$ as usual). Note that $S^*(f_i)(a_j) = f_i(S(a_j)) = e^{(j)}_i$ so $S^*(f_i) = \sum_j e^{(j)}_i f_j$.

Since H is graded and connected we also know that H° is graded and connected, so S° is uniquely defined by satisfying

$$\cdot^\circ(S^\circ \otimes \mathrm{Id})\Delta^\circ = u^\circ \circ \epsilon^\circ$$

Consider this with S^* on the dual basis. For $f_0 = \mathrm{Id}$ we have

$$\mathrm{Id} = u^\circ(\epsilon^\circ(f_0))$$

and

$$\cdot^\circ(S^* \otimes \mathrm{Id})\Delta^\circ(f_0) = S^*(\mathrm{Id}) = \mathrm{Id}$$

as desired. For $i > 0$ we have

$$0 = u^\circ(\epsilon^\circ(f_i))$$

and

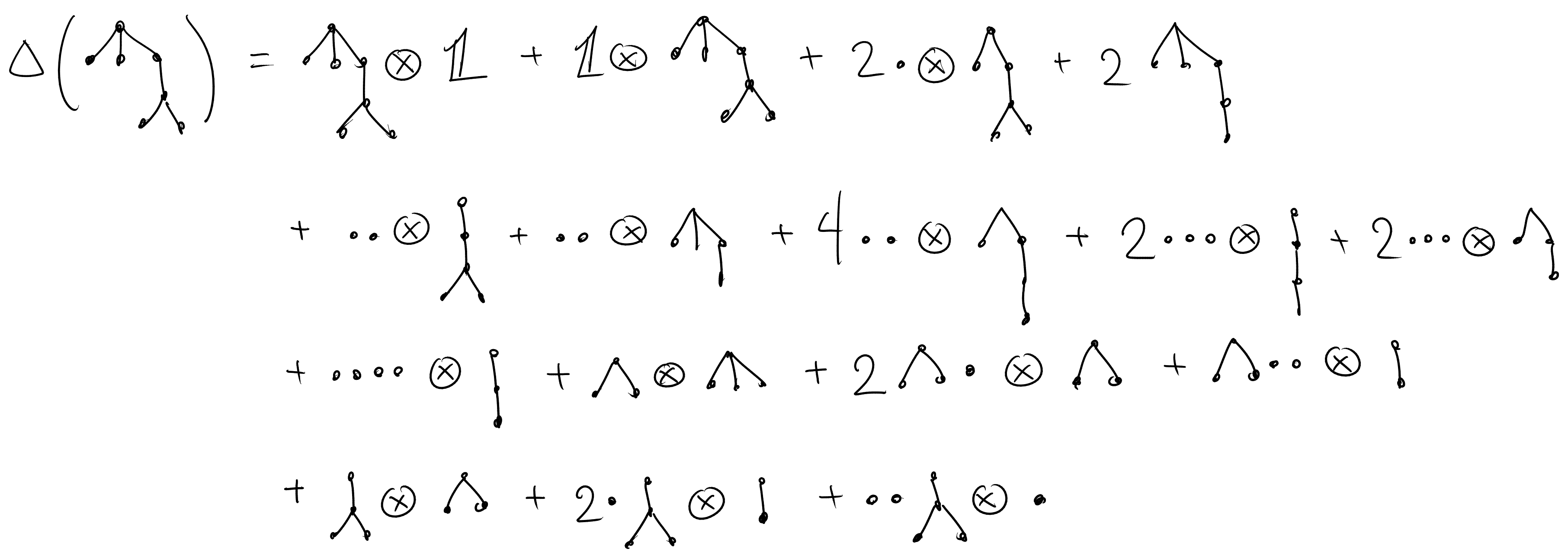
$$\cdot^\circ(S^* \otimes \mathrm{Id})\Delta^\circ(f_i) = \sum_{j,k,\ell,m} c^{(m)}_{\ell,k} e^{(\ell)}_j d^{(i)}_{j,k} f_m = 0$$

by (1), which proves the result.

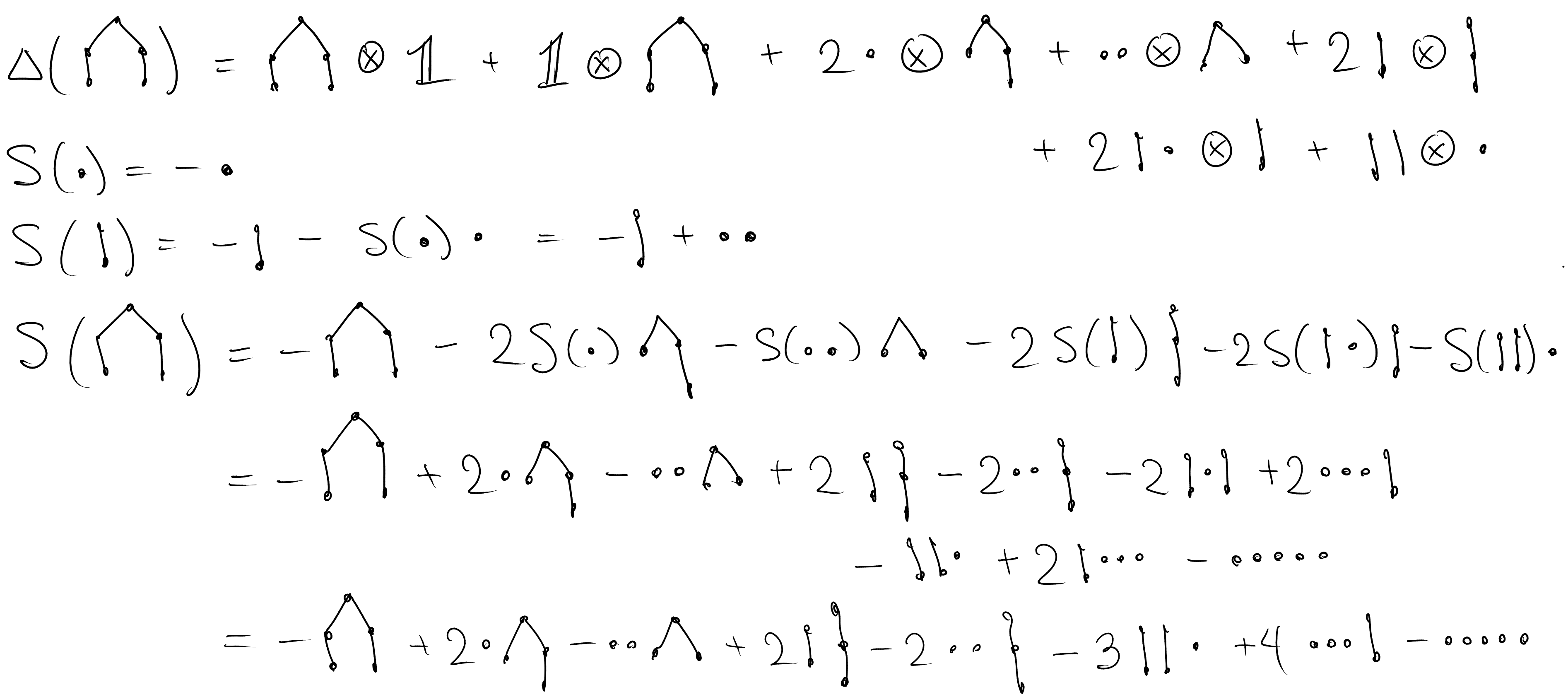
(4) See handwritten pages appended.

(5) See `http://loic.foissy.free.fr/pageperso/preprint3.pdf`

a)



b)



c) A basis for all elements of order 3 is $\mathfrak{l}_3, \Lambda, \bullet\mathfrak{l}_2, \bullet\bullet\bullet$

calculate the nonprimitive part of the coproducts of each

$$\tilde{\Delta}(\mathfrak{l}_3) = \bullet \otimes \mathfrak{l}_2 + \mathfrak{l}_2 \otimes \bullet$$

$$\tilde{\Delta}(\Lambda) = 2\bullet \otimes \mathfrak{l}_2 + \bullet\bullet \otimes \bullet$$

$$\tilde{\Delta}(\bullet\mathfrak{l}_2) = \bullet \otimes \mathfrak{l}_2 + \bullet\bullet \otimes \bullet + \mathfrak{l}_2 \otimes \bullet + \bullet \otimes \bullet\bullet$$

$$\tilde{\Delta}(\bullet\bullet\bullet) = 3\bullet \otimes \bullet\bullet + 3\bullet\bullet \otimes \bullet$$

Every linear combination of these equaling 0 gives a primitive element and every primitive element can be built in this way

No such linear combination can involve $\tilde{\Delta}(\Lambda)$ since the other three are all cocommutative but it is not. By inspection the only nontrivial linear combinations of the others giving 0 are

$$3a\,\tilde{\Delta}(\mathfrak{l}_3) + a\,\tilde{\Delta}(\bullet\bullet\bullet) - 3a\,\tilde{\Delta}(\bullet\mathfrak{l}_2), \quad a \in k^*$$

So the primitive elements of degree 3 are

$$3a\,\mathfrak{l}_3 + a\bullet\bullet\bullet - 3a\bullet\mathfrak{l}_2, \quad a \in k^*$$