Math 821, Spring 2013, Lecture 10

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Proposition 1. *Suppose a is a k-vector space which is both a k-algebra under $(\cdot, u)$ and a k-coalgebra under (Δ, ϵ). The following are equivalent;*

1. *(Δ, ϵ) are algebra homomorphisms (ie. A is a bialgebra)*
2. *$(\cdot, u)$ are coalgebra morphisms*
3. *the four diagrams in Figure 1 commute.*

$$\begin{tikzcd} k \arrow[rr, "id"] \arrow[dr, "u"'] & & k \\ & A \arrow[ur, "\epsilon"'] & \end{tikzcd}$$

(a)

$$\begin{tikzcd} & A \otimes A \arrow[dl, "\Delta \otimes \Delta"'] \arrow[dd, "\cdot"] \\ A \otimes A \otimes A \otimes A \arrow[d, "id \otimes \tau \otimes id"'] & \\ A \otimes A \otimes A \otimes A \arrow[dr, "\cdot \otimes \cdot"'] & A \arrow[d, "\Delta"] \\ & A \otimes A \end{tikzcd}$$

(b)

$$\begin{tikzcd} A \otimes A \arrow[r, "\epsilon \otimes \epsilon"] \arrow[d, "\cdot"'] & k \otimes k \arrow[d, "\cdot"] \\ A \arrow[r, "\epsilon"] & k \end{tikzcd}$$

(c)

$$\begin{tikzcd} k \arrow[r, "u"] \arrow[d, "\Delta"'] & A \arrow[d, "\Delta"] \\ k \otimes k \arrow[r, "u \otimes u"] & A \otimes A \end{tikzcd}$$

(d)

Figure 1

Proof. ($1 \Leftrightarrow 3$) the fact that (Δ, ϵ) are algebra homomorphisms immediately implies (3), purely from the definition, and vice versa.

($2 \Leftrightarrow 3$) Reversing the arrows and relabelling accordingly; $\cdot \leftrightarrow \Delta$ and $\epsilon \leftrightarrow u$, (c) and (d) switch and both (a) and (b) remain the same. By duality then, the statements (2) and (3) are equivalent. $\square$

Example. Bialgebra of words; Tensor algebra (abbr. TA)

We can make the tensor algebra $TV = \bigoplus_{n=0}^{\infty} V^{\otimes n}$ into a bialgebra via

$$\epsilon\mid_{V^{\otimes 0}} = id,$$

$$\epsilon\mid_{\bigoplus_{n=0}^{\infty} V^{\otimes n}} = 0,$$

and

$$\Delta(a) = a \otimes \mathbf{1} + \mathbf{1} \otimes a$$

for $a \in A$, and extend as an algebra homomorphism.

For example, writing $\Delta(a \otimes b \otimes c)$ as $\Delta(abc)$,

$$\begin{aligned}
\Delta(abc) &= \Delta(a)\Delta(b)\Delta(c) \\
&= (a \otimes \mathbf{1} + \mathbf{1} \otimes a)(b \otimes \mathbf{1} + \mathbf{1} \otimes b)(c \otimes \mathbf{1} + \mathbf{1} \otimes c) \\
&= abc \otimes \mathbf{1} + ab \otimes c + ac \otimes b + a \otimes bc \\
&\qquad + bc \otimes a + b \otimes ac + c \otimes ab + \mathbf{1} \otimes abc.
\end{aligned}$$

So, this is a kind of pulling apart operation. We can pull out all possible subwords, putting the subword on the left and the remaining part on the right. This is an *antishuffle*.

We now check that this structure behaves the way we want it to.

If $a \in A$,

$$\begin{aligned}
(id \otimes \Delta)(\Delta(a)) &= (id \otimes \Delta)(a \otimes \mathbf{1} + \mathbf{1} \otimes a) \\
&= a \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes a + \mathbf{1} \otimes a \otimes \mathbf{1} \\
(\Delta \otimes id)(\Delta(a)) &= (\Delta \otimes id)(a \otimes \mathbf{1} + \mathbf{1} \otimes a) \\
&= \mathbf{1} \otimes a \otimes \mathbf{1} + a \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes a
\end{aligned}$$

If $w_1, w_2 \in TA$,

$$\begin{aligned}
(id \otimes \Delta)(\Delta w_1 w_2) &= (id \otimes \Delta)(\Delta w_1 \Delta w_2) \\
&= ((id \otimes \Delta)\Delta w_1)((id \otimes \Delta)\Delta w_2) \\
&= (\Delta \otimes id)\Delta w_1((\Delta \otimes id)\Delta w_2 \\
&= (\Delta \otimes id)\Delta(w_1 w_2).
\end{aligned}$$

$$\begin{tikzcd}
TA \otimes k \arrow[r] & TA & k \otimes TA \arrow[l] \\
TA \otimes TA \arrow[u, "id \otimes \epsilon"] & & TA \otimes TA \arrow[u, "\epsilon \otimes id"'] \\
& TA \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] \arrow[uu, "id"'] &
\end{tikzcd}$$

Figure 2

For the counit property, we want TA to behave as in Figure 2.
Take a word $w = a_1 a_2 \cdots a_n \in TA$. Then,

$$\Delta(w) = w \otimes \mathbf{1} + \mathbf{1} \otimes w + k$$

where $k \in (\bigoplus_{n=1}^{\infty} A^{\otimes n})\,(\bigoplus_{n=1}^{\infty} A^{\otimes n})$. So $(\epsilon \otimes id)\Delta(w) = \mathbf{1} \otimes w \mapsto w$ and $(id \otimes \epsilon)\Delta(w) = w \otimes \mathbf{1} \mapsto w$.

More Vocabulary

Definition. A k-algebra A is *commutative* if it acts as in Figure 3, and *cocommutative* if it acts as in Figure 4.

$$\begin{tikzcd}
A \otimes A \arrow[rr, "\tau"] \arrow[dr, "\cdot"] & & A \otimes A \arrow[dl, "\cdot"'] \\
& A &
\end{tikzcd}$$

Figure 3

$$\begin{tikzcd}
C \otimes C & & C \otimes C \arrow[ll, "\tau"'] \\
& C \arrow[ul, "\cdot"] \arrow[ur, "\cdot"'] &
\end{tikzcd}$$

Figure 4

Example. The tensor algebra is not commutative, as multiplication is concatenation. It is cocommutative, however.

Example. Recall the Connes-Kreimer Hopf algebra of rooted trees from last class. It is commutative, as multiplication is disjoint union. It is not cocommutative, though, as there may be forests on the left, but only trees on the right.

Definition. A *graded k-vector space* is a k-vector space with a direct sum decomposition

$$V = \bigoplus_{k=0}^{\infty} V_n.$$

Call the elements of V_n *homogeneous of degree n.*

Note that if V, W are graded k-vector spaces, then $V \otimes W$ is also graded via

$$(V \otimes W)_n = \bigoplus_{k=0}^{n} V_k \otimes W_{n-k}.$$

Definition. A linear map $f : V \to W$ between graded vector spaces is *graded* (of degree zero) if $f(V_n) \subseteq W_n$ for all n.

Definition. An algebra, coalgebra, or bialgebra is *graded* if the underlying vector space is graded and the defining maps $(\cdot, u, \Delta, \epsilon)$ are graded.

Example. We can demonstrate k is a graded bialgebra in the trivial way; put everything in degree zero.

Example. The tensor algebra is graded by length of words.

Example. The Connes-Kreimer algebra is graded by number of vertices.

In general
Take a combinatorial class $\mathcal{C}$. Form a graded vector space

$$V = \bigoplus_{k=0}^{\infty} V_k,$$

where $V_k = \operatorname{span}(\mathcal{C}_k)$. So, $V = V\mathcal{C}$, the vector space given by $\mathcal{C}$.

Then, find an interesting "putting together" map for multiplication and "taking apart" map for comultiplication.

Often, multiplication will be disjoint union if we have connected and not connected objects, and coproduct will be about pulling out subobjects.

Definition. A graded k-vector space is called *connected* if $V_0 \cong k$.

Example. The tensor algebra is connected by definition.

Example. The Connes-Kreimer algebra is connected.

Combinatorially, if $c_0 = 1$, so $\mathcal{C} = \{\mathbf{1}\}$, a very typical situation, then the degree zero part of $V\mathcal{C}$ is $\mathrm{span}\{\mathbf{1}\} \cong k$. Hence, connected is very natural.

In graded connected bialgebras we get a lot of this for free.

Proposition 2. *Let A be a graded connected bialgebra over k.*

1. *$u : k \to A_0$ is an isomorphism*
2. *$\epsilon|_{A_0} : A_0 \to k$ is the reverse isomorphism*
3. $\ker \epsilon = \bigoplus_{n=1}^{\infty} A_n$
4. *for $x \in \ker \epsilon$, $\Delta(x) = \mathbf{1} \otimes x + x \otimes \mathbf{1} + \tilde{\Delta}(x)$ where $\tilde{\Delta}(x) \in \ker \epsilon \otimes \ker \epsilon$*

Proof. (1) As u is a graded map, $u(k) \subseteq A_0$ and $\dim_k k = \dim_k A_0 = 1$. Further, u is one-to-one since, if $u(\lambda_1) = u(\lambda_2)$, we know that this behaves as in Figure 5.

$$\begin{tikzcd}
A_0 \otimes k \arrow[d, "id \otimes u"'] & A_0 \arrow[l] \arrow[r] \arrow[dd, leftrightarrow, "id"] & k \otimes A_0 \arrow[d, "u \otimes id"] \\
A_0 \otimes A_0 \arrow[dr, "\cdot"] & & A_0 \otimes A_0 \arrow[dl, "\cdot"'] \\
& A_0 &
\end{tikzcd}$$

Figure 5

Then, as $(id \otimes u)(\mathbf{1} \otimes \lambda_1) = (id \otimes u)(\mathbf{1} \otimes \lambda_2)$, for $i \in \{1, 2\}$

$$\begin{aligned}
(id \otimes u)(\mathbf{1} \otimes \lambda_i) &= (id \otimes u)(\lambda_i \otimes \mathbf{1}) \\
&= \lambda_i \otimes \mathbf{1},
\end{aligned}$$

and therefore $\lambda_1 = \lambda_2$.

(2) From (1), we know that Figure 6 holds. Thus, $\epsilon|_{A_0}$ is the reverse isomorphism.

$$\begin{tikzcd}
k \arrow[rr, "id"] \arrow[dr, "u"'] & & k \\
& A \arrow[ur, "\epsilon"'] &
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
k \arrow[rr, "id"] \arrow[dr, "u"'] & & k \\
& A_0 \arrow[ur, "\epsilon"'] &
\end{tikzcd}$$

Figure 6

(3) As ϵ is a graded map but k exists only in degree zero,

$$\epsilon\left(\bigoplus_{n=1}^{\infty} A_n\right) = 0$$

and by (2) nothing else maps to zero, so $\ker \epsilon = \bigoplus_{n=1}^{\infty} A_n$.

(4) Note Figure 7.

$$\begin{tikzcd}
A \otimes k \arrow[r] & A & k \otimes A \arrow[l] \\
A \otimes A \arrow[u, "id \otimes \epsilon"] & A \arrow[l, "\Delta"'] \arrow[r, "\Delta"'] \arrow[u, "id_A"'] & A \otimes A \arrow[u, "\epsilon \otimes id"']
\end{tikzcd}$$

Figure 7

Take $x \in \ker \epsilon$ by the right path through the diagram. Then,

$$\Delta(x) = \lambda_1 \otimes \frac{1}{\lambda_1} x + j = \mathbf{1} \otimes x + j$$

where $j \in (\ker \epsilon) \otimes A$. Similarly, taking $x \in \ker \epsilon$ by the left path through the diagram,

$$\Delta(x) = x \otimes \mathbf{1} + q$$

where $q \in A \otimes \ker \epsilon$. Thus,

$$\Delta(x) = x \otimes \mathbf{1} + \mathbf{1} \otimes x + \tilde{\Delta}(x)$$

where $\tilde{\Delta}(x) \in \ker \epsilon \otimes \ker \epsilon$. $\square$

Definition. Suppose A is a bialgebra and $x \in A$. If $\Delta(x) = \mathbf{1} \otimes x + x \otimes \mathbf{1}$ we say that x is *primitive*. If $\Delta(x) = x \otimes x$ we say that x is *group-like*. When $\Delta(x) = \mathbf{1} \otimes x + x \otimes \mathbf{1} + \tilde{\Delta}(x)$, we say that $\mathbf{1} \otimes x + x \otimes \mathbf{1}$ is the *primitive part*.

Combinatorially, u and ϵ are just two more ways of looking at $\mathcal{C}_0 = \{\mathbf{1}\}$.

References: Victor Reiner's notes, 1.1-1.3.