Math 821, Spring 2013, Lecture 2

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1 Generating Functions

Last time we defined $\mathcal{A} = \mathcal{B} \times \mathcal{C}$, $\mathcal{A} = \mathcal{B} + \mathcal{C}$, $\mathcal{E}$, $\mathcal{Z}$.

Notation. Let $\mathcal{B}$ be a combinatorial class. Write $\mathcal{B}^n = \underbrace{\mathcal{B} \times \ldots \times \mathcal{B}}_{n \text{ times}}$, for $n > 0$, and $\mathcal{B}^0 = \mathcal{E}$.

Definition. Let $\mathcal{B}$ be a combinatorial class with $\mathcal{B}_0 = \varnothing$. Define $\mathcal{A} = Seq(\mathcal{B})$ to be the combinatorial class with underlying set $\bigcup_{n=0}^{\infty} \mathcal{B}^n$ (note that the union is disjoint), and the size function is defined as follows: Take $a \in \mathcal{A}$, then $a \in \mathcal{B}^n$, for some $n \geq 0$. Let $|a|_{\mathcal{A}} = |a|_{\mathcal{B}^n}$.

Another way to write this is to say that $Seq(\mathcal{B})$ is combinatorically isomorphic to $\mathcal{E} + \mathcal{B} + \mathcal{B}^2 + \ldots$, where $\mathcal{E}$ is the empty sequence, $\mathcal{B}$ sequence of length 1, $\mathcal{B}^2$ sequence of length 2, and so on, except that infinite combinatorial sum has not been defined (see above definition). So $Seq(\mathcal{B})$ is the class of all finite sequences of elements of $\mathcal{B}$.

Note. Why did we require $\mathcal{B}_0 = \varnothing$ in the definition above? Because if $\varepsilon \in \mathcal{B}_0$ then $(\ ), (\varepsilon), (\varepsilon, \varepsilon), (\varepsilon, \varepsilon, \varepsilon), \ldots$ are all elements of size 0 in $Seq(\mathcal{B})$, so $Seq(\mathcal{B})$ would not be a combinatorial class.

Proposition 1. *Let $\mathcal{B}$ be a combinatorial class with $\mathcal{B}_0 = \varnothing$, and $\mathcal{A} = Seq(\mathcal{B})$. Then $A(x) = 1 + B(x) + B(x)^2 + B(x)^3 + \ldots = \frac{1}{1-B(x)}$.*

Note. The above infinite sum is a well-defined infinite sum of formal power series, since $\mathcal{B}_0 = \varnothing$, so $b_0 = 0$, so $B(x)^k = b_k^k x^k +$ higher order terms, so to find

$$[x^n](1 + B(x) + B(x)^2 + \ldots)$$

we only need the finite sum

$$[x^n](1 + B(x) + \ldots + B(x)^n).$$

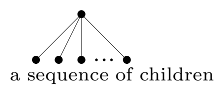
Proof of Proposition 1.

$$a_n = |\mathcal{A}_n| = |(\bigcup_{i=0}^{n} \mathcal{B}^i)_n|, \text{ since } \mathcal{B}_0 = \varnothing$$
$$= [x^n](1 + B(x) + \ldots + B(x)^n) = [x^n]\left(\frac{1}{1 - B(x)}\right), \text{ since } B(0) = 0.$$

□

Example. Let $\mathcal{D}$ be the class of Dyck Paths. Last time we saw the decomposition $\mathcal{D} = Seq(\mathcal{Z}_{\nearrow} \times \mathcal{D} \times \mathcal{Z}_{\searrow})$. So we can read right off this $D(x) = \frac{1}{1-x^2 D(x)}$.

Example. *Rooted plane trees* A rooted plane tree could be ε, or



. Call the class of these trees $\mathcal{P}$, so $\mathcal{P} = \mathcal{E} + \mathcal{Z} \times Seq(\mathcal{P} - \varepsilon)$, or for nonempty plane trees $\mathcal{N}$, $\mathcal{N} = \mathcal{Z} \times Seq(\mathcal{N})$. And so immediately we obtain $P(x) = 1 + x \cdot \frac{1}{1-(P(x)-1)} = 1 + x \cdot \frac{1}{2-P(x)}$, or $N(x) = \frac{x}{1-N(x)}$.

Example. *Binary strings* $\mathcal{B} = Seq(\mathcal{Z}_0 + \mathcal{Z}_1)$, so $B(x) = \frac{1}{1-2x}$.

Example. *Binary strings with no two consecutive 0s* Such a string can be presented as $(\varepsilon + 0)(1^*10)^*1^*$, so we have $\mathcal{B} = (\mathcal{E} + \mathcal{Z}_0) \times Seq(\mathcal{Z}_1 \times Seq(\mathcal{Z}_1) \times \mathcal{Z}_0) \times Seq(\mathcal{Z}_1)$, therefore

$$B(x) = (1 + x)\left(\frac{1}{1 - \left(\frac{x^2}{1-x}\right)}\right)\left(\frac{1}{1-x}\right) = \frac{1+x}{1-x-x^2}.$$

Example. *Integers*≥ 1 $\mathcal{I} = Seq(\mathcal{Z})$, so $I(x) = \frac{1}{1-x}$.

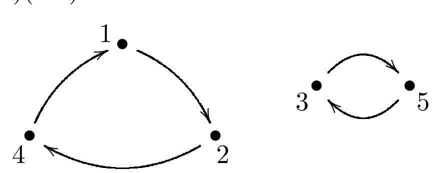
2 Polya Cycle Index Polynomials

Definition. Let S_n be the group of permutations of $\{1, 2, \ldots, n\}$. A subgroup of S_n will be called a permutation group.

Proposition 2. *Let $\sigma \in S_n$, then σ can be written uniquely as a product of disjoint cycles.*

We can use cycle notation to represent permutations.

Example. (1 2 4)(3 5)



Definition. Let A be a permutation group on $\{1, 2, \ldots, n\}$. For $\sigma \in S_n$ let $j_k(\sigma)$ be the number of cycles of length k in the disjoint cycle representation of σ. Then the *cycle index polynomial* of A in variables $s_1, \ldots, s_n$ is $Z(A; s_1, s_2, \ldots, s_n) = \frac{1}{|A|} \sum_{\sigma \in A} \left(\prod_{k=1}^{n} s_k^{j_k(\sigma)} \right)$.

Example. $A = S_3 = \{(1)(2)(3), (1)(2\ 3), (2)(1\ 3), (3)(1\ 2), (1\ 2\ 3), (1\ 3\ 2)\}$. So $Z(S_3; s_1, s_2, s_3) = \frac{1}{6}(s_1^3 + 3s_1 s_2 + 2s_3)$.

Definition. Let $\mathcal{C}$ be a combinatorial class, and $X = \{1, \ldots, n\}$. Then $\mathcal{C}^X$ is the combinatorial class whose underlying set is the set of functions $f : X \to \mathcal{C}$ with size function $|f| = \sum_{i=1}^{n} |f(i)|$, i.e $\mathcal{C}^X$ is combinatorically isomorphic to $\mathcal{C}^n$.

If A is a permutation group on X, then $\mathcal{C}^X/A$ is the combinatorial class whose underlying set is equivalence classes of elements of $\mathcal{C}^X$ under the relation

$$f_1 \sim f_2 \text{ iff } \exists \sigma \in A \text{ such that } f_1 \circ \sigma = f_2.$$

Then $|f_1| = |f_2|$ in $\mathcal{C}^X$, so $\mathcal{C}^X/A$ inherits this size function.

Theorem 3. *Polya Enumeration Theorem*
Let $\mathcal{C}$ be a combinatorial class, $X = \{1, \ldots, n\}$, A be a permutation group on X, and $\mathcal{B} = \mathcal{C}^X/A$. Then $B(x) = Z(A; C(x), C(x^2), \ldots, C(x^n))$.

To prove the theorem first we need the following definition:

Definition. Let A be a permutation group on $X = \{1, \ldots, n\}$. For $x \in X$, the *orbit* of x is $\{y \in X : \ \exists \sigma \in A, \sigma x = y\}$.
Alternately view A as giving an equivalence relation on X via

$$x \sim y \text{ iff } \exists \sigma \in A, \sigma x = y,$$

then the orbits are the equivalence classes.

Proposition 4. *Burnside's Lemma*
let A be a permutation group on $X = \{1, \ldots, n\}$. Then the number of orbits of A, $N(A)$, is $N(A) = \frac{1}{|A|} \sum_{\sigma \in A} j_1(\sigma)$.

Note. This says that the number of orbits is the average number of fixed points.

Proof. For $x \in X$, let $Stab_A(x) = \{\sigma \in A : \ \sigma x = x\}$ (the stabilizer of x). Let Y be an orbit of A, and take $y \in Y$. We have a bijection between Y and the cosets of $Stab_A(y)$. To see this let $\sigma_1 Stab_A(y), \ldots, \sigma_n Stab_A(y)$ be the cosets. Then the map

$$\sigma_i Stab_A(y) \longmapsto \sigma_i(y)$$

is a bijection. The map is one-to-one, as if $\sigma_i(y) = \sigma_j(y)$ then $\sigma_j^{-1}\sigma_i(y) = y$, so $\sigma_j^{-1}\sigma_i \in Stab_A(y)$, so σ_i and σ_j are in the same coset. The map is also onto, as for any $u \in Y$ there is an $\alpha \in A, \alpha y = u$. But the union of the cosets is A, so $\alpha = \sigma_i \alpha'$, so $\sigma_i(y) = \sigma_i(\alpha' y) = \alpha y = u$, since $\alpha' \in Stab_A(y)$.

Now just count. From the bijection we know that

$$|Stab_A(y)||Y| = |A|. \qquad (*)$$

Now sum this over all the orbits $X_1, \ldots, X_{N(A)}$:

$$\sum_{i=1}^{N(A)} |Stab_A(x_i)|\ |X_i| = N(A)|A|, \text{ where } x_i \in X_i.$$

Note that if x and y are in the same orbit then $|Stab_A(x)| = |Stab_A(y)|$, because of $(*)$. So

$$\begin{aligned} N(A)|A| &= \sum_{i=1}^{N(A)} |Stab_A(x_i)||X_i| \\ &= \sum_{x \in X} |Stab_A(x)| \\ &= \sum_{x \in X} \sum_{\sigma \in Stab_A(x)} 1 \\ &= \sum_{\sigma \in A} \sum_{\substack{x \in X \\ \sigma x = x}} 1 \\ &= \sum_{\sigma \in A} j_1(\sigma) \end{aligned}$$

□

Proof (of Polya enumeration theorem). Let $b_k(\alpha) = |\{b \in \mathcal{C}^X : |b| = k, b \circ \alpha = b\}|$. Then Burnside's lemma says $b_k = \frac{1}{|A|} \sum_{\sigma \in A} b_k(\sigma)$, so

$$\begin{aligned} B(x) &= \sum_{k=0}^{\infty} \frac{1}{|A|} \sum_{\sigma \in A} b_k(\sigma) x^k \\ &= \frac{1}{|A|} \sum_{\sigma \in A} \sum_{k=0}^{\infty} b_k(\sigma) x^k, \end{aligned}$$

where $\sum_{k=0}^{\infty} b_k(\sigma) x^k$ is the generating function for elements of $\mathcal{C}^X$ fixed by σ. Let b be such an element, i.e an element of $\mathcal{C}^X$ with $b \circ \sigma = b$. Then $b(i)$ is constant for all i in any given cycle of σ. So

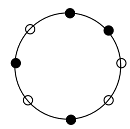
$$\begin{aligned} \sum_{k=0}^{\infty} b_k(\sigma) x^k &= \prod_{\substack{\text{cycles } \alpha \\ \text{in } \sigma}} \left(\sum_{i=1}^{\infty} c_i x^{|\alpha| i} \right) \\ &= \prod_{k=1}^{n} \left(\sum_{i=1}^{\infty} c_i x^{k_i} \right)^{j_k(\sigma)} \\ &= \prod_{k=1}^{n} C(x^k)^{j_k(\sigma)} \end{aligned}$$

So

$$B(x) = \frac{1}{|A|}\sum_{\sigma\in A}\prod_{k=1}^{n} C(x^k)^{j_k(\sigma)}$$
$$= Z(A; C(x), C(x^2), \ldots, C(x^n)).$$

□

Example. *Binary necklaces*
Let $\mathcal{N}$ be the combinatorial class of cycles as graphs with two colours of vertices.



First consider size 4:

$$\mathcal{N}_4 = (\mathcal{Z}_\circ + \mathcal{Z}_\bullet)^{\{1,2,3,4\}}/D_4,$$

where D_n is the Dihedral group of order n. Note that if we do not allow flipping then we would have $(\mathcal{Z}_\circ + \mathcal{Z}_\bullet)^{\{1,2,3,4\}}/C_4$, where C_n is the cyclic group of order n.

$$D_4 = \{(1)(2)(3)(4), (1)(3)(2\ 4), (2)(4)(1\ 3)\},$$
$$(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3), (1\ 2\ 3\ 4), (1\ 4\ 3\ 2)\}$$

So

$$N_4(x) = Z(D_4; 2x, 2x^2, 2x^3, 2x^4)$$
$$= \frac{1}{8}((2x)^4 + 2(2x)^2(2x^2) + 3(2x^2)^2 + 2(2x^4))$$

References. Harary, F. and Palmer, E. M. , *Graphical Enumeration*, Academic Press (1973). Ch. 2.