Variants of the Kernel Method for Lattice Path Models

by

Stephen Ryan Melczer

B.Sc. (Hons.), Simon Fraser University, 2012

A Thesis submitted in partial fulfillment of the requirements for the degree of

Master of Science

in the

Department of Mathematics

Faculty of Science

© Stephen Ryan Melczer 2014
SIMON FRASER UNIVERSITY
Summer 2014

All rights reserved.

However, in accordance with the Copyright Act of Canada, this work may be reproduced without authorization under the conditions for “Fair Dealing.” Therefore, limited reproduction of this work for the purposes of private study, research, criticism, review and news reporting is likely to be in accordance with the law, particularly if cited appropriately.
APPROVAL

Name: Stephen Ryan Melczer

Degree: Master of Science

Title of Thesis: Variants of the Kernel Method for Lattice Path Models

Examining Committee:
Dr. Jonathan Jedwab
Professor
Chair

Dr. Marni Mishna
Associate Professor
Senior Supervisor

Dr. Michael Monagan
Professor
Supervisor

Dr. Andrew Rechnitzer
Associate Professor
Department of Mathematics
University of British Columbia
External Examiner

Date Approved: May 5th, 2014
Partial Copyright Licence

The author, whose copyright is declared on the title page of this work, has granted to Simon Fraser University the non-exclusive, royalty-free right to include a digital copy of this thesis, project or extended essay[s] and associated supplemental files (“Work”) (title[s] below) in Summit, the Institutional Research Repository at SFU. SFU may also make copies of the Work for purposes of a scholarly or research nature; for users of the SFU Library; or in response to a request from another library, or educational institution, on SFU’s own behalf or for one of its users. Distribution may be in any form.

The author has further agreed that SFU may keep more than one copy of the Work for purposes of back-up and security; and that SFU may, without changing the content, translate, if technically possible, the Work to any medium or format for the purpose of preserving the Work and facilitating the exercise of SFU’s rights under this licence.

It is understood that copying, publication, or public performance of the Work for commercial purposes shall not be allowed without the author’s written permission.

While granting the above uses to SFU, the author retains copyright ownership and moral rights in the Work, and may deal with the copyright in the Work in any way consistent with the terms of this licence, including the right to change the Work for subsequent purposes, including editing and publishing the Work in whole or in part, and licensing the content to other parties as the author may desire.

The author represents and warrants that he/she has the right to grant the rights contained in this licence and that the Work does not, to the best of the author’s knowledge, infringe upon anyone’s copyright. The author has obtained written copyright permission, where required, for the use of any third-party copyrighted material contained in the Work. The author represents and warrants that the Work is his/her own original work and that he/she has not previously assigned or relinquished the rights conferred in this licence.

Simon Fraser University Library
Burnaby, British Columbia, Canada

revised Fall 2013
Abstract

The kernel method has proved to be an extremely versatile tool for exact and asymptotic enumeration. Recent applications in the study of lattice walks have linked combinatorial properties of a model to algebraic conditions on its generating function, demonstrating how to extract additional information from the process. This thesis details two new results. In the first, we apply the iterated kernel method to determine asymptotic information about a family of models in the quarter plane, finding their generating functions explicitly and classifying them as non D-finite. The second considers $d$-dimensional walks restricted to an octant whose step sets are symmetric over every axis. A generalized version of the orbit sum method allows for a representation of their generating functions as diagonals of multivariate rational functions, proving they are D-finite. In combination with current developments from analytic combinatorics in several variables, this yields dominant asymptotics for all such models.
To my parents, for all their love and support.
Acknowledgments

I would like to thank the following people and institutions for all their help and support during the past couple of years:

Alin Bostan and Manuel Kauers for inviting me over to Europe (my first real trip outside of Canada!), supervising me during that time, and continuing to be amazing collaborators;

Mireille Bousquet-Mélou and Bruno Salvy for their sage wisdom, advise, and mentoring;

Inria (and Microsoft Research) for sponsoring my original internship in Paris, and Flavia Stan, Thomas Sibut-Pinote, Frédéric Chyzak, Christoph Koutschan, and Pierre Lairez for making me feel welcome so far from home;

Nav, Adrian, Avi, Sam, and all my other friends who’ve helped to make this the best time of my life;

Michael Monagan and Andrew Rechnitzer for their role on this jury;

The Natural Sciences and Engineering Research Council of Canada, the Math Department at SFU, SFU at an institutional level, and the Office for Science and Technology at the Embassy of France in Canada for supporting the research conducted during this degree;

Marni Mishna for introducing me to the topics which have come to dominate my academic life, for keeping me busy and on my toes, and for whipping this thesis into its current state (also, for all the trips to France). You gave me opportunities I could have never dreamed of before we started working together.
Contents

Approval ii

Partial Copyright License iii

Abstract iv

Dedication v

Acknowledgments vi

Contents vii

1 Introduction 1
   1.1 Overview of Topics ........................................... 4
   1.2 Original Contributions ........................................ 5

2 Generating Functions 6
   2.1 Multivariate Formal Power Series and Laurent Series .......... 7
   2.2 Rational Generating Functions .................................. 8
   2.3 Algebraic Generating Functions .................................. 9
   2.4 D-Finite Generating Functions .................................. 11
   2.5 Diagonals of Formal Power Series ............................. 13

3 Lattice Path Models 18
   3.1 Unrestricted Lattice Walks ................................... 19
   3.2 Lattice Walks in a Halfspace ................................... 21
   3.3 Lattice Walks in the Quarter Plane ............................ 24
## 3.3.1 The Kernel Equation and Group of a Walk

3.3.2 Diagonals of Rational Functions

3.3.3 A Computer Algebra Approach

## 4 Non D-Finite Models in the Quarter Plane

4.1 The Iterated Kernel Method

4.1.1 Find the roots of the kernel.

4.1.2 Determine an explicit expression for $C(x, 0, t)$.

4.1.3 Exhibit distinct singularities of $\mathcal{Y}_n$.

4.1.4 Prove that $C(t)$ is non D-finite.

4.2 The Non-Symmetric Models

4.3 Other Methods for Non D-Finiteness

## 5 Highly Symmetric Walks in an Orthant

5.1 The Orbit Sum Method in Higher Dimensions

5.1.1 A Functional Equation

5.1.2 The Orbit Sum

5.1.3 The Generating Function as a Diagonal

5.1.4 The Singular Variety

5.2 Analytic Combinatorics in Several Variables

5.2.1 Critical Points

5.2.2 Minimal Points

5.2.3 Asymptotics Results

## 6 Generalizations

6.1 Non Symmetric Step Sets in an Orthant

6.1.1 ACSV in the General Case

6.2 Walk Models in an Octant

6.2.1 Two Dimensional Walks

6.2.2 Three Dimensional Walks

6.3 Walks With Long Steps

## 7 Conclusion

Bibliography
Chapter 1

Introduction

The kernel method is an important algebraic tool for studying generating functions satisfying functional equations. Although the method has been used in many forms across many different subject areas, and has thus been rediscovered many times, accounts of its beginnings often mention the following exercise from the popular 1968 textbook of Knuth:

Exercise 1 (Ex. 4, Section 2.2.1, Knuth [44]). Consider a word composed of \( n \) ‘S’ symbols and \( n \) ‘X’ symbols, where S stands for “add an element” to some specified stack and X stands for “remove an element” from that stack. Such a word is called admissible if it specifies no operations that cannot be performed – i.e., if the number of X’s never exceeds the number of S’s when read from left to right. Find the number of admissible words as a function of \( n \).

This problem is not modern – indeed, counting the number of admissible words is equivalent to the well known ballot problem originating in the early eighteenth century – and Knuth first gives a proof using the classical reflection principle from the nineteenth century. However, he continues on to write:

“We present here a new method for solving the ballot problem with the use of double generating functions, since this method lends itself to the solution of more difficult problems…”

– Knuth [44]

Letting \( g_{n,m} \) denote the number of admissible sequences of S’s and X’s of (combined) length \( n \) such that there are \( m \) more S’s than X’s, he uses a recurrence on \( g_{n,m} \) to derive a functional equation for the bi-variate generating function \( G(x, z) = \sum_{n,m \geq 0} g_{n,m} x^m z^n \). This
functional equation, now known as the kernel equation, allowed Knuth to express $G(x, z)$ explicitly in terms of the power series solutions of a polynomial coefficient in the equation and thus determine $g_{n,k}$. Well known examples of the kernel method which helped to modernize and develop it as a distinct strategy of proof include Bousquet-Mélou and Petkovšek [20], Banderier et al. [4], Bousquet-Mélou [17], and Janse van Rensburg et al. [40]. The kernel method also shares many similarities with the ‘Bethe Ansatz’ method widely used in physics, introduced by Bethe [7] in 1931 to find eigenvalues and eigenvectors of a one dimensional statistical physics model.

The focus of this thesis is the use of the kernel method and recent variants in the study of lattice path models – for examples see Figures 1.1 and 1.2 – in particular for walks restricted to the sub-lattice $\mathbb{N}^d$ of $\mathbb{Z}^d$, known as walks restricted to an orthant. When the dimension $d$ equals one such walks are called walks restricted to a halfspace, and when the dimension is two they are known as walks restricted to a quarter plane. There is a strong connection between these models and the kernel method; in fact, the ballot problem mentioned by Knuth asks one to count the number of walks on a restricted integer lattice.

**Example 2** (The ballot problem). Consider the lattice path model in one dimension whose elements take steps in $S = \{-1, 1\} \subset \mathbb{Z}$, must stay in the natural numbers, and end at the integer $k$. This is known as the ballot problem, as it models the number of ways to count votes in an election between two candidates A and B where A wins by $k$ votes and never trails as the votes are tallied one by one.

![Figure 1.1: Two unrestricted lattice walks of length 50.](image_url)
As laid out in the historical survey of Humphreys [38], the earliest accounts of what are now considered lattice path problems arose in probabilistic contexts as far back as the seventeenth century studies of Pascal and Fermat, including examples analogous to the ballot problem in the work of de Moivre [53] in 1711. An 1878 work of Whitworth [62] uses explicit lattice path terminology (for instance “paces” from an origin) to consider “Arrangements of \( m \) things of one sort and \( n \) things of another sort under certain conditions of priority”, and answered questions posed by the Educational Times in 1878 including the probability of drinking \( n \) glasses of wine and \( n \) glasses of water in a random order while never drinking more wine than water.

Lattice walks in the early twentieth century were considered by many to be a recreational topic – best exemplified by an article of Grossman [37] entitled “Fun with lattice points” and published in the journal Scripta Mathematica aimed at the layperson. The mid twentieth century saw strong interest in lattice walks and the related topic of random walks from the field of physics. Lattice path models are able to model physical phenomena through their application to statistical mechanics, for instance in the study of polymers in a solution (see Janse van Rensburg [39]). Modern applications include results in statistical mechanics, formal language theory, queuing theory (see Böhm [8]), the analysis of data structures, and the study of other combinatorial structures such as plane partitions (see Alegri et al. [1]), to name a small sample among many.
1.1 Overview of Topics

Chapter 2 of this thesis contains background material on formal power series and generating functions. It outlines results from the study of univariate and multivariate power series and Laurent series, followed by a detailed discussion on rational, algebraic, and D-finite power series. Applications of these classifications are shown (both in this chapter and in the rest of the thesis), as is the connection between the members of each class under the operation of extracting a diagonal.

Chapter 3 defines lattice path models and exhibits how the kernel method is used to enumerate both unrestricted walks and those restricted to lie in a halfspace. It also begins a survey on the classification of lattice walks restricted to a quarter plane, exhibiting two kernel method variants for proving D-finiteness when it arises. Chapter 4 concludes the material on walks in a quarter plane by describing in detail an approach – called the iterated kernel method – for proving that a family of quarter plane models admit non D-finite generating functions, along with determining their asymptotics.

Chapter 5 shows how to generalize the orbit sum variant of the kernel method to study walks in any dimension which are restricted to an orthant and have unit step sets with a symmetry across each axis. This generalization allows one to represent the generating function of a model as the diagonal of a multivariate rational function, thus proving it is D-finite. In addition, combining this diagonal representation with new results in the study of analytic combinatorics in several variables gives the dominant asymptotics explicitly for every such model.

Chapter 6 discusses ongoing work extending the above results. The variant of the kernel method discussed in Chapter 5, when combined with results from analytic combinatorics in several variables, should allow for a more general analysis of D-finite models missing some symmetries, however an increase in the geometric complexity of algebraic varieties arising in the course of the method creates complications which much be overcome. A systematic survey of lattice walks in three dimensions restricted to the positive octant which is currently underway by Bostan, Bousquet-Mélou, Kauers, and Melczer [13] is also detailed, as is an ongoing study of walks in a quarter plane taking non-unit steps.

Finally, Chapter 7 lists some of the major open problems in the study of D-finite functions and restricted lattice path models.
CHAPTER 1. INTRODUCTION

1.2 Original Contributions

The main original work contained in this thesis is the following:

- The results of Chapter 4, which extend previous work of Mishna and Rechnitzer [52] to prove non D-finiteness for all singular walks in the quadrant (precise details on which results are original are contained in the chapter). An extended abstract of this work was published in the proceedings of FPSAC 2013 and a full manuscript (Melczer and Mishna [50]) was accepted into a special issue of Combinatorics, Probability and Computing dedicated to the memory of Philippe Flajolet.

- All results in Chapter 5 are original and together constitute the main contribution of this thesis. An extended abstract of this work has been accepted into the proceedings of the 25th International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms (AofA’14) to be held in Paris during June 2014. A full manuscript is currently in preparation.

- The ongoing work listed in Chapter 6 is original, most notably the survey on three dimensional lattice walks confined to an octant with Bostan, Bousquet-Mélou, and Kauers [13].
Chapter 2

Generating Functions

We begin with background material on generating functions and formal power series. Given a ring \( R \), we let \( R[[z]] \) denote the usual ring of formal power series (see Stanley [61] or Wilf [63] for well known treatments of the relevant background material). Although the elements of \( R[[z]] \) are only series in a formal sense, there is a bounty of useful theory available for formal series which represent real or complex analytic functions in a neighbourhood of the origin.

With this in mind, the formal derivative of \( F(z) \in R[[z]] \) is defined by

\[
\frac{d}{dz} F(z) := \sum_{n=1}^{\infty} n f_n z^{n-1},
\]

so that if the series defining \( F(z) \) represents an analytic function in some neighbourhood of the origin, then the series for the formal derivative represents the series for the analytic derivative of this analytic function near the origin.

Formal power series connect to combinatorics through the notions of combinatorial classes and generating functions. A combinatorial class is a countable (or finite) set of objects \( C \) together with a function \( |\cdot| : C \to \mathbb{N} \) such that for any \( n \in \mathbb{N} \) there are a finite number of elements of \( C \) — called the elements of size \( n \) — that map to \( n \) under \( |\cdot| \). We let \( C_n \) denote the set containing the elements of size \( n \), \( c_n = |C_n| \), and call \((c_n)_{n=0}^{\infty}\) the counting sequence of the class.

Given a sequence \((s_n)_{n=0}^{\infty}\), the generating function of the sequence is

\[
S(z) = \sum_{n \geq 0} s_n z^n,
\]
and if \((s_n)\) is the counting sequence of a combinatorial class \(S\) we say that \(S(z)\) is the generating function of \(S\). By definition, the generating function of a class is a member of \(\mathbb{N}[[z]]\), but we can also consider it as a member of \(R[[z]]\) for any ring \(R\) containing \(\mathbb{N}\) (for instance \(R = \mathbb{Q}\) or \(R = \mathbb{C}\)).

Dating all the way back to their origins in the eighteenth century work of de Moivre [28], generating functions have been incredibly useful because they give a framework for formal manipulations of counting sequences. Furthermore, combinatorial problems often yield generating functions which represent complex analytic functions. In this situation one can derive combinatorial properties of a class from the analytic properties of its generating function (here we abuse language and say that the generating function, which is a formal power series, is also the analytic function which it represents at the origin). For instance, the dominant asymptotics of a counting sequence which is not entire is determined by the location and nature of its singularities closest to the origin (called its dominant singularities).

We now examine how classifying generating functions according to their algebraic properties allows one to give combinatorial meaning to those properties, along with asymptotic results on the corresponding classes. We talk about the classes of rational functions, algebraic functions, and D-finite functions, each less restrictive than the last. Finally, we discuss the diagonal operator on formal power series in multiple variables and examine its interaction with members of each class.

### 2.1 Multivariate Formal Power Series and Laurent Series

In many combinatorial applications we wish to track some parameter (or set of parameters) along with the size of an object; for this, it is useful to consider multivariate formal power series. Given a ring \(R\) and some positive integer \(n\), we define the ring of formal power series in the variables \(z = (z_1, \ldots, z_n)\) recursively as \(R[[z]] = R[[z_1, \ldots, z_n]] := R[[z_1]][[z_2]] \cdots [[z_n]]\).

An element \(F(z) \in R[[z]]\) has the form

\[
F(z_1, \ldots, z_n) = \sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} f_{i_1, \ldots, i_n} z_1^{i_1} \cdots z_n^{i_n},
\]

where \(f_{i_1, \ldots, i_n} \in R\) for \(0 \leq i_1, i_2, \ldots, i_n < \infty\). This inherits many properties from the ring of formal power series in one variable – in particular \(R[[z]]\) is an integral domain whenever \(R\) is an integral domain, and \(F(z)\) is invertible in \(R[[z]]\) if and only if \(f_{0,0,\ldots,0}\) is invertible in
R. For convenience, if \( i = (i_1, \ldots, i_n) \) then we define \( z^i := z_1^{i_1} \cdots z_n^{i_n} \), and for \( c \in \mathbb{N} \) we let \( z^c := z^{(c, \ldots, c)} \). We also extend the derivative to the multivariate setting by defining

\[
\frac{\partial}{\partial z_k} F(z) := \sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} i_k \cdot f_{i_1, i_2, \ldots, i_n} z_1^{i_1} \cdots z_k^{i_k-1} \cdots z_n^{i_n}.
\]

As \( K[[z]] \) is an integral domain for every field \( K \) we may wish to find its field of fractions, called the field of formal Laurent series and denoted \( K((z)) \). When working with one variable, any non-zero \( F(z) \in K[[z]] \) can be written \( F(z) = z^k G(z) \), where \( G(z) \in K[[z]] \) is invertible. The formal element \( z^{-k} G(z)^{-1} = \sum_{n \geq -k} g'_n z^n \) then satisfies \( f \cdot (z^{-k} g^{-1}) = 1 \) when the multiplication is extended in the obvious way from the multiplication in \( K[[z]] \).

Thus, we take \( K((z)) \) to be is the field whose elements have the form \( \sum_{n \geq -k} g'_n z^n \) for some \( k \geq 0 \). One way to construct the field of formal Laurent series in \( n \) variables – which we adopt here – is recursively through \( K((z)) := K((z_1))((z_2)) \cdots ((z_n)) \). There are other ways of constructing the field of Laurent series and the reader is directed to Aparicio-Monforte and Kauers [2] for a more complete discussion.

2.2 Rational Generating Functions

Given a formal power series \( A(z) \), we say that \( A(z) \) is a polynomial if there are only a finite number of terms in the series representation of \( A(z) \) which are non-zero. Any element \( F(z) \) in \( K[[z]] \) which can be written as \( A(z)/B(z) \), for polynomials \( A(z) \) and \( B(z) \) with \( B(z) \) having a non-zero constant term (so its multiplicative inverse is defined as a formal power series) is called a rational formal power series. We let \( K[z] \) and \( K(z) \) denote the set of polynomials and rational power series over \( K \), respectively. To lighten notation, in the univariate case we define \( \overline{z} := z^{-1} \) and let \( K[\overline{z}, z] \) be the subring of \( K((z)) \) – called the Laurent polynomials – which have a finite number of terms with non-zero coefficients.

Given \( r \in \mathbb{N} \), a sequence \( (f_n)_{n=0}^{\infty} \) of elements of the field \( K \) is called a linear recurrence relation of order \( r \) with constant coefficients if

\[
f_{n+r} = c_{r-1} f_{n+r-1} + c_{r-2} f_{n+r-2} + \cdots + c_0 f_n, \quad c_i \in K, \quad c_0 \neq 0 \quad (2.1)
\]

for all \( n \geq 0 \). Rational generating functions play an important role in the study of recurrences due to the following result.
Theorem 3. Suppose that \((f_n)_{n=0}^\infty\) is a linear recurrence relation with constant coefficients satisfying equation (2.1) above. Then the generating function \(F(z) = \sum_{k\geq 0} f_k z^k\) is a rational function
\[
F(z) = \frac{A(z)}{1-(c_{r-1}z + c_{r-2}z^2 + \cdots + c_0 z^r)},
\]
where the highest non-zero coefficient of the polynomial \(A(z)\) has index less than \(r\).

Proof. This follows from the calculation
\[
[z^{n+r}] \left( 1 - (c_{r-1}z + c_{r-2}z^2 + \cdots + c_0 z^r) \right) \left( \sum_{k\geq 0} f_k z^k \right) = f_{n+r} - \sum_{m=0}^{r-1} c_m f_{n+r-m} = 0.
\]

Since a rational function in the complex plane is analytic in the entire plane minus the zero set of its denominator, we see that the generating function for a linear recurrence with constant coefficients is the Taylor series of a rational function in some disk centered at the origin. Combined with a partial fraction decomposition, this also gives asymptotic results.

Theorem 4. Suppose that \((f_n)_{n=0}^\infty\) is a linear recurrence relation with constant coefficients satisfying equation (2.1) above, and let \(\alpha_1, \ldots, \alpha_m\) be the distinct roots of the polynomial \(P(z) = 1-(c_{n-1}z + \cdots + c_0 z^r)\) in the complex plane. Then there exist polynomials \(\{P_j(z)\}_{j=1}^m\) such that for all \(n\) larger than some fixed \(n_0\),
\[
f_n = \sum_{j=1}^m P_j(n) \alpha_j^{-n}.
\]
The degree of the polynomial \(P_j(z)\) is one less than the order of the zero of \(P(z)\) at \(z = \alpha_j\).

Example 5. Crucial to the development of the theory of rational functions has been the study of regular languages over an alphabet, as the generating function for the number of words of length \(n\) in a regular language will always be rational (see Chapter 6 of Stanley [60] for details on this and the inverse problem of determining when a rational power series can be interpreted as the number of words of length \(n\) in some rational language).

2.3 Algebraic Generating Functions

Let \(K\) be an algebraically closed field. A formal power series \(F(z) \in K[[z]]\) is called algebraic if there exist polynomials \(p_0(z), \ldots, p_d(z) \in K[z]\), not all zero, such that
\[
p_d(z) F(z)^d + p_{d-1}(z) F(z)^{d-1} + \cdots + p_0(z) = 0.
\]
CHAPTER 2. GENERATING FUNCTIONS

Given algebraic $F(z)$, the monic polynomial $m(z, y) \in K[z][y]$ of minimal degree in $y$ such that $m(z, F(z)) = 0$ is called the minimal polynomial of $F(x)$. Conversely, given polynomial $m(z, y) \in K[z][y]$ it is natural to wonder if there is a formal power series or formal Laurent series which satisfies $m(z, F(z)) = 0$. In general, the answer is no.

**Example 6.** Let $m(z, y) = 1 + zy^2$, and towards a contradiction suppose that there exists $F(z) = \sum_{n \geq k} f_n z^n \in \mathbb{C}((z))$, with $k \in \mathbb{Z}$ and $f_k \neq 0$, such that $m(z, F) = 0$. If $k \geq 0$ then the lowest power of $z$ to appear in the power series $zF(z)^2$ is at least one, so $m(z, F(z))$ would have a constant term of one, not zero. If $k < 0$ then expanding gives

$$0 = m(z, F(z)) = 1 + z \left( f_k^2 z^{2k} + \sum_{n \geq 2k+1} f_n' z^n \right).$$

Since $2k + 1 < 0$, this would imply $f_k^2 = 0$ which contradicts $f_k \neq 0$ and $K$ a field.  

In the univariate case, when $m(z, y)$ admits a formal power series solution which is the Taylor series of a complex valued function at the origin, one can use the theory of Newton-Puiseux expansions to determine the asymptotic form of the series coefficients.

**Theorem 7** (Flajolet and Sedgewick [34], Theorem VII.8). Let $f(z) = \sum_{n \geq 0} f_n z^n$ be an algebraic function which is analytic in the complex plane at zero, and let $\beta, \omega_1 \beta, \ldots, \omega_{m-1} \beta$ be the singularities of $f(z)$ on its circle of convergence (so $|\omega_j| = 1$ for each $j$). Then

$$f_n = \frac{\beta^{-n} n^s}{\Gamma(s+1)} \cdot \sum_{i \geq 0} c_i \omega_i^m + O(\beta^{-n} n^t), \quad (2.2)$$

where $s \in \mathbb{Q}$ is not a negative integer, $s > t$, and the $c_i$ are algebraic.

**Example 8.** Just as regular languages helped give a key interpretation of rational generating functions, Chomsky and Schützenberger [22] showed that the sequence counting the number of words of length $n$ in any unambiguous context-free language has an algebraic generating function. This was used by Flajolet [33], along with the asymptotic result found in Theorem 7, to prove the ambiguity of certain context-free languages.  

**Example 9.** It was shown by Banderier and Flajolet [3] that the generating function for the number of integer lattice walks in the plane which take steps from some finite set and never leave the positive half plane is always algebraic. The details of this result are described in Section 3.2.
2.4 D-Finite Generating Functions

Given an element \( F(z) \in K[[z]] \), we call \( F(z) \) **D-finite (of order \( r \) and degree \( d \))** if there exist polynomials \( a_0(z), \ldots, a_r(z) \) whose highest non-zero coefficients have indices at most \( d \) such that

\[
a_0(z)F(z) + a_1(z)\frac{d}{dz}F(z) + \cdots + a_r(z)\frac{d^r}{dz^r}F(z) = 0, \quad a_r(z) \neq 0. \tag{2.3}
\]

Alternatively, one could define \( F(z) \) to be D-finite if \( F(z) \) and its partial derivatives \( \frac{d^k}{dz^k}F(z) \) form a finite dimensional vector space over \( K(z) \). Indeed, we call multivariate \( F(z) \in K[[z]] \) D-finite if \( F(z) \) and its partial derivatives \( \partial^{i_1 + \cdots + i_n}/\partial z_1^{i_1} \cdots \partial z_n^{i_n} \) span a finite dimensional vector space over \( K(z) \).

Just like the case of rational functions, D-finite functions are important to the study of liner recurrences. Given positive integer \( r \), a sequence \( (f_n)_{n=0}^{\infty} \) of elements of the field \( K \) is called a **linear recurrence relation with polynomial coefficients (of order \( r \) and degree \( d \))** if

\[
c_r(n)f_{n+r} = c_{r-1}(n)f_{n+r-1} + c_{r-2}(n)f_{n+r-2} + \cdots + c_0(n)f_n, \quad c_0(z), c_r(z) \neq 0 \tag{2.4}
\]

for all \( n \geq 0 \), where the degree of each \( c_i(z) \in K[z] \) is at most \( d \); such a sequence is also called **holonomic**.

**Theorem 10** (Theorem 7.1, Kauers and Paule [42]). Let \( F(z) = \sum_{n \geq 0} f_n z^n \in K[[z]] \).

1. If \( F(z) \) is D-finite of order \( r \) and degree \( d \), then \( (f_n)_{n=0}^{\infty} \) is holonomic of order at most \( r + d \) and degree at most \( r \).

2. If \( (a_n)_{n=0}^{\infty} \) is holonomic of order \( r \) and degree \( d \), then \( F(z) \) is D-finite of order at most \( d \) and degree at most \( r + d \).

**Proof.** If \( F(z) = \sum_{n \geq 0} f_n z^n \in K[[z]] \) then

\[
z^i \frac{d^m}{dz^m} F(z) = \sum_{n=i}^{\infty} (n - i + m)(n - i + (m - 1)) \cdots (n - i + 1) f_{n-i+m} z^n. \tag{2.5}
\]

The statements follow from equations (2.3) and (2.4) through this correspondence. \( \square \)

The study of D-finite functions in a combinatorial context was popularized by Stanley [59], who surveyed many examples and closure properties (see also Stanley [60]). Later work of Zeilberger [64] continued this interest by providing automatic means of proving identities on holonomic sequences; this area of research is still extremely active (for example Chyzak [25] and Bostan et al. [12]).
Example 11. Let $H_n = \sum_{k=1}^{n} \frac{1}{k}$ be the sequence of harmonic sums. Then $H_n = H_{n-1} + \frac{1}{n}$, so $nH_n = nH_{n-1} + 1$ for $n \geq 1$. This implies

$$nH_n = nH_{n-1} + 1 \quad \text{and} \quad (n-1)H_{n-2} + 1 = (n-1)H_{n-1},$$

so that $nH_n = (2n-1)H_{n-1} - (n-1)H_{n-2}$ when $n \geq 2$. Using the correspondence in Equation (2.5) gives

$$(1 - z)^2 \frac{d}{dz} H(z) - (1 - z)H(z) - 1 = 0,$$

so that

$$(1 - z)^2 \frac{d^2}{dz^2} H(z) - 3(1 - z) \frac{d}{dz} H(z) + H(z) = 0,$$

where $H(z)$ is the generating function $H(z) = \sum_{n \geq 0} H_n z^n$. Solving this ODE with the correct initial conditions tells us that the power series $H(z)$ is the Taylor series representation of the analytic function $H(z) = \frac{1}{1-z} \log \frac{1}{1-z}$ in the open unit disk, even though there is no closed form for $H_n$.

Given a differential equation satisfied by some D-finite function, one can determine asymptotic results of its coefficients and generate them extremely efficiently. In this way linear differential equations with polynomial coefficients provide a nice data structure for dealing with many generating functions, just as minimal polynomials provide a convenient data structure for algebraic numbers (or generating functions) in computer algebra systems.

Theorem 12 (Theorem 15, Bostan [9]). Suppose $K = \mathbb{Z}$ and $a_r(0) \neq 0$ in Equation (2.3). Then Equation (2.3) admits a power series solution whose the first $N$ coefficients can be calculated in $O(\max(d, r) r \log(r)^2(1 + N/r))$ integer operations.

In order to classify a generating function as D-finite, the following properties are often useful. We see more examples of proving and disproving D-finiteness in the later chapters of this thesis.

Theorem 13. Let $A(z)$ be D-finite.

1. The set of D-finite functions in $K[[z]]$ is a sub-algebra of $K[[z]]$. Moreover, given annihilating differential equations of D-finite $F(z)$ and $G(z)$ one can calculate explicit annihilating differential equations of $F(z)G(z)$ and $c_1F(z) + c_2G(z)$ for $c_1, c_2 \in K$.

2. If $B(z)$ is algebraic then $B(z)$ is D-finite.
3. If \( B(z) \) is algebraic and \( B(0) = 0 \) then \( A(B(z)) \) is D-finite.

4. \( \frac{d}{dz} A(z) \) and \( \int_z A(z) \) are D-finite.

5. If \( A(z) \) is the Taylor series of an analytic function in a region then it has a finite number of singularities in that region, given by the roots of \( a_r(z) \) in Equation (2.3).

Proof. These results and references can be found in Chapter 6 of Stanley [60] and Chapter 7 of Kauers and Paule [42].

One problem of great interest is to give a combinatorial interpretation to the class of D-finite generating functions. While regular languages correspond in some sense to rational generating functions (Example 5), and context-free languages correspond to algebraic generating functions (Example 8), there is no simple correspondence between D-finite functions and a well known formal language class. As seen in the upcoming chapters of this thesis, lattice path models in restricted regions give many interesting examples of generating functions lying in this class (although such models do not always give rise to D-finite generating functions).

### 2.5 Diagonals of Formal Power Series

Given a power series

\[
F(z_1, \ldots, z_n) = \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} f_{i_1, i_2, \ldots, i_n} z_1^{i_1} \cdots z_n^{i_n}
\]

in \( R[[z]] \), the (complete) diagonal of \( F(z) \), denoted \( (\Delta F)(z) \), is defined by

\[
(\Delta F)(z_n) = \sum_{k=0}^{\infty} f_{k,k,\ldots,k} z_n^k.
\]

That is, the diagonal is the formal power series whose non-zero terms are the terms of \( F(z) \) with all indices being equal. We also define the partial diagonal in the indices \( i \) and \( j \) to be the formal power series \( \Delta_{i,j}F \in R[[z_1, \ldots, z_i-1, z_{i+1}, \ldots, z_n]] \) obtained by taking all terms with the indices of \( z_i \) and \( z_j \) being equal, and then ignoring the variable \( z_i \) by specializing it to 1.

One of the fundamental notions related to diagonals is how it affects the various classes of generating functions discussed above.
Theorem 14 (Hautus-Klarner-Polya-Furstenberg). Let $F(x, y) = P(x, y)/Q(x, y) \in \mathbb{C}[[z]]$ be a rational power series in two variables. Then $\Delta F$ is algebraic.

Proof. As $F(x, y)$ is rational it represents the Taylor series of an analytic function in some neighbourhood of the origin, thus for $|y|$ sufficiently small the complex valued function $F(x, y/x)$ in the variable $x$ uniformly converges in an annulus $A_{|y|}$ around the origin. Let $C_{|y|}$ be a positively oriented circle around the origin staying in $A_{|y|}$. Then the Cauchy Integral Formula implies

$$
(\Delta F)(y) = \frac{1}{2\pi i} \int_{C_{|y|}} \frac{P(x, y/x)}{xQ(x, y/x)} \, dx = \sum_{i=1}^{n} \text{Res} \left( \frac{P(x, y/x)}{xQ(x, y/z)} ; x = \rho_i \right),
$$

(2.6)

where $\rho_1(y), \ldots, \rho_n(y)$ are the roots of $Q(x, y/x)$ which converge to 0 as $y$ approaches 0 (so that they are the poles of the integrand which stay inside all contours $C_{|y|}$ as $y$ approaches the origin). Thus, as each residue is algebraic and the sum of algebraic functions is algebraic, $\Delta F$ is algebraic. \qed

There is also a formal proof of this result using Newton-Puiseux expansions (see Stanley [60]).

Example 15. Let

$$
F(x, y) = \frac{1}{1 - x - y} = \sum_{k=0}^{\infty} (x + y)^k = \sum_{(i_1, i_2) \in \mathbb{N}^2} \binom{i_1 + i_2}{i_1} x^{i_1} y^{i_2},
$$

so that

$$
(\Delta F)(y) = \sum_{n=0}^{\infty} \binom{2n}{n} y^n.
$$

Then for a fixed $|y| < 1$, the series for $F(x, y)$ converges uniformly in the disk $|x| < r$, where $0 < r < 1 - |y|$. Thus, in a sufficiently small annulus around the origin the function $F(x, y/x)$ is analytic in $x$ and for $C$ a circle staying in this annulus,

$$
(\Delta f)(y) = \frac{1}{2\pi i} \int_{C} \frac{dx}{x - x^2 - y}.
$$

This rational function has poles at $x = \frac{1 \pm \sqrt{1 - 4y}}{2}$, and only the negative branch of the square root converges to the origin (the other branch converges to 1 and thus will not be contained in $C$ for $y$ sufficiently small). This implies

$$
\sum_{n=0}^{\infty} \binom{2n}{n} y^n = (\Delta f)(y) = \text{Res} \left( \frac{1}{x - x^2 - y} ; x = \frac{1 - \sqrt{1 - 4y}}{2} \right) = \frac{1}{\sqrt{1 - 4y}}.
$$

\fi
This method can be automated (Chyzak and Salvy [26]), and has been implemented in
the Maple packages gfun and Mgfun, however the diagonal of a rational function in more
than two variables may not be algebraic (see Example 17). Although the class of rational
functions is not closed under taking diagonals, a result of Lipshitz shows that the class of
D-finite functions is closed under this operation. In this sense, the D-finite functions are a
closure of the rational functions under the operations of taking diagonals (Christol [23] was
the first to show that the diagonal of a rational function is always D-finite).

**Theorem 16** (Lipshitz [49]). If $F(z) \in K[[z]]$ is D-finite, then $\Delta_{i,j}F(z)$ is D-finite for all
$1 \leq i < j \leq n$. In particular, the complete diagonal $\Delta F$ is D-finite.

**Example 17.** Let

$$F(z_1, z_2, z_3, z) = \frac{1}{1 - z_1 - z_2} \cdot \frac{1}{1 - z_3 - z}$$

$$= \sum_{(i_1, \ldots, i_4) \in \mathbb{N}^4} \binom{i_1 + i_2}{i_1} \binom{i_3 + i_4}{i_3} z_{1}^{i_1} z_{2}^{i_2} z_{3}^{i_3} z^{i_4}.$$

Then

$$(\Delta F)(z) = \sum_{n \geq 0} \binom{2n}{n} 2 z^n,$$

which is not algebraic since Stirling’s approximation implies that its coefficients grow asymptotically as $8^n/(\pi n^3)$, violating the constraints given by Theorem 7. However, binomial
identities imply that the coefficients of the diagonal satisfy the recurrence relation

$$n^2 F_n - 4(4n^2 - 4n + 1)F_{n-1} = 0,$$

so $(\Delta F)(z)$ satisfies the differential equation

$$(z - 16z^2) \frac{d^2}{dz^2} y(z) + (1 - 32z) \frac{d}{dz} y(z) - 4y(z) = 0.$$

Conversely, it is also of interest (and great application, as we see in later chapters) to
know when a function which belongs to one of the above classes can be written as the
diagonal of a function belonging to a simpler class. In particular, there has been much work
on determining when a function can be represented as the diagonal of a rational function.
Theorem 18 (Furstenberg [35], Denef and Lipshitz [29]). Let $A$ be an integral domain and $P(y, z) \in A[y, z]$ with $(\partial P/\partial y)(0, 0)$ a unit in $A$. If $F(z) \in A[[z]]$ has a zero constant term and $P(F(z), z) = 0$ then

$$F(z) = \Delta \left( \frac{y^2(\partial P/\partial y)(y, zy)}{P(y, zy)} \right),$$

i.e., $F$ is the diagonal of a rational function in two variables.

Proof. Writing $P(y, z) = (y - F(z))g(y, z)$ for $g \in A[[z]][y]$ we have

$$(\partial P/\partial y)(y, z) = g(y, z) + (y - F(z))(\partial g/\partial y)(y, z).$$

Thus,

$$\frac{y^2(\partial P/\partial y)(y, zy)}{P(y, zy)} = \frac{y^2}{y - F(zy)} + \frac{y^2(\partial g/\partial y)(y, zy)}{g(y, zy)}.$$

Although it may not appear to be,

$$\frac{y^2}{y - F(zy)} = \frac{y}{1 - F(zy)/y}$$

is a power series as $F(z)$ has a zero constant term, and applying the diagonal operator yields $F(z)$. Furthermore, the second summand in Equation (2.7) is a power series by our assumption on $(\partial P/\partial y)(0, 0)$ and has a diagonal of zero. The result follows from the distributivity of the diagonal operator over addition.

In fact, the property of being a diagonal of a rational function continues to hold when the minimal polynomial of $F(z)$ has vanishing derivative at the origin, and extends further to the case of several variables (although the result is no longer explicit).

Theorem 19 (Denef and Lipshitz [29], Theorem 6.2). Let $F(z) \in K[[z_1, \ldots, z_n]]$ be an algebraic power series, where $K$ is a field (or, in more generality, an excellent local integral domain). Then there exists a rational power series $R(z_1, \ldots, z_n, z_{n+1}, \ldots, z_{2n})$ in the ring $K[[z_1, \ldots, z_{2n}]]$ such that

$$F(z) = \Delta_{1,n+1}\Delta_{2,n+2} \cdots \Delta_{n,2n}R(z_1, \ldots, z_{2n}).$$

One may ask whether a similar result holds for D-finite functions; indeed, such a result was conjectured by Christol in 1990 for globally bounded D-finite functions and remains open. A power series $F(z) \in \mathbb{C}[[z]]$ is globally bounded if $F(z)$ represents the Taylor series of an analytic function in some neighbourhood of the origin, and there exist $a, b \in \mathbb{Q}$ such that $aF(bz)$ has integer coefficients.
Conjecture 20 (Christol [24], Conjecture 4). Every $D$-finite globally bounded function is the diagonal of a rational function.

We conclude this chapter with one important application of diagonals. Given power series $A(z) = \sum_{i \in \mathbb{N}} a_i z^i$ and $B(z) = \sum_{i \in \mathbb{N}} b_i z^i$, the Hadamard product of $A(z)$ and $B(z)$ is the power series $A(z) \circ B(z) = \sum_{i \in \mathbb{N}} a_i b_i z^i$ obtained by multiplying term by term (so called ‘naive multiplication’). Since their origins, diagonals were used to study the Hadamard product of power series (see Cameron and Martin [21]) due the relations

\begin{align*}
A(z) \circ B(z) &= \Delta_{1,n+1} \Delta_{2,n+2} \cdots \Delta_{n,2n} A(z_1, \ldots, z_n) B(z_{n+1}, \ldots, z_{2n}), \quad (2.8) \\
\Delta_{1,2} A(z) &= \left. \left( A(z) \circ \frac{1}{1-z_1 z_2} \cdot \prod_{j=3}^{n} \frac{1}{1-z_j} \right) \right|_{z_1=1}, \quad (2.9)
\end{align*}

which can be verified by direct computation. Thus, all of the closure results above have analogous statements for the Hadamard product.
Chapter 3

Lattice Path Models

A lattice path model is a combinatorial class which encodes the number of ways to “move” on a lattice subject to certain constraints. More precisely, given a dimension $d \in \mathbb{N}$, a finite set of allowable steps $S \subseteq \mathbb{Z}^d$, and a restricting region $R \subseteq \mathbb{Z}^d$ the integer lattice path model taking steps in $S$ restricted to $R$ is the combinatorial class consisting of sequences of the form $(s_1, \ldots, s_k)$, where $s_j \in S$ for $1 \leq j \leq k$ and every partial sum $s_1 + \cdots + s_r \in R$ for $1 \leq r \leq k$ (addition is performed in the vector space $\mathbb{Z}^d$; i.e., component-wise).

The size of an element in this class is the length of the sequence (the number of steps it contains), and by convention we add a single sequence of length zero representing an empty walk. We view such a sequence as a path or walk starting at the origin in $\mathbb{Z}^d$ which successively takes steps from $S$ and always stays in the region $R$ by drawing line segments between the endpoints of the partial sums of the sequence. We may also restrict the class further by adding other constraints, for instance only admitting sequences which end in some terminal set $T \subseteq \mathbb{Z}^d$ (the element sum of each sequence in the class lies in $T$).

Although there is a rich theory of walks on other lattices, here we consider only walks on the integer lattice in various dimensions. We begin by discussing models whose walks are unrestricted, which always have rational generating functions, followed by models whose walks are restricted to a halfplane, which always have algebraic generating functions. Finally, we consider models whose walks are restricted to a quarter plane, which can have transcendental and even non D-finite generating functions. Strategies for proving the D-finiteness of quarter plane generating functions are given in this chapter, while kernel method variants for proving non D-finiteness are discussed in Chapter 4.
3.1 Unrestricted Lattice Walks

Consider first a lattice path model $C$ with step set $S = \{s_1, \ldots, s_m\} \subseteq \mathbb{Z}^d$ and $R = \mathbb{Z}^d$, so that there is no restriction on where the walks in the model can move. As a walk of length $n$ can have any of the $|S|$ steps in each of its $n$ coordinates, we have the generating function identity

$$C(t) = \sum_{n \geq 0} |S|^n t^n = \frac{1}{1 - |S|t}. \quad (3.1)$$

Note also that the generating function is analytic in the neighbourhood $|t| < 1/|S|$ of the origin, and in fact admits $t = 1/|S|$ as its only singularity.

Although unrestricted models are simple to enumerate, we use them as an opportunity to set up the basics of the kernel method. Instead of looking at the univariate generating function enumerating the total number of walks in the model, the key of the method is to use a multivariate generating function to additionally keep track of each walk’s endpoint. With this in mind, we define the formal power series in $t$ with coefficients in $\mathbb{Q}((z))$

$$C(z,t) := \sum_{n \in \mathbb{N}} \left( \sum_{i \in \mathbb{Z}^d} c_{i,n} z^i \right) t^n,$$

where $c_{i,n}$ denotes the number of walks of length $n$ which end at the point $i \in \mathbb{Z}^d$. As there are a finite number of walks of a given length, this formal series actually lives in the subring $\mathbb{Q}[z_1, z_2, \ldots, z_d][[t]] \subseteq \mathbb{Q}((z))[[t]]$. Let

$$P(z) = \sum_{i \in S} z^i,$$

which is called the characteristic polynomial of the model, and define

$$C_n(z) := \sum_{i \in \mathbb{Z}^d} c_{i,n} z^i,$$

for each $n \geq 0$. Combinatorially, a walk of length $n + 1$ is a walk of length $n$ followed by a step in $S$. Updating the endpoint of a walk appropriately, we get the recurrence

$$C_0(z) = 1, \quad C_{n+1}(z) = P(z)C_n(z) \quad \text{for } n \geq 0, \quad (3.1)$$

which, upon multiplying by $t^{n+1}$ and summing, yields the expression

$$(1 - P(z))C(z,t) = 1. \quad (3.2)$$
Equation (3.2) is called the kernel equation, with the Laurent polynomial \( K(z) = 1 - P(z) \) known as the kernel. In the following sections we show that the kernel equation can be set up in a similar manner for many lattice path models, however its right hand side is more complex, relying on evaluations and coefficient extractions of the (a priori unknown in explicit form) multivariate generating function. In this simple case we can simply solve the kernel equation to obtain

\[
C(z, t) = \sum_{n \geq 0} C_n(z)t^n = \sum_{n \geq 0} P(z)^n t^n = \frac{1}{1 - tP(z)},
\]

as an element of \( \mathbb{Q}[z_1, z_1, \ldots, z_d, z_d][[t]] \). Note that \( C(1, t) = C(t) \), the univariate generating function above, and that as \( P(z) \) is bounded in sufficiently small neighbourhoods of \( z = 1 \), the formal series \( C(z, t) \) represents a (multivariate) analytic function in a neighbourhood of \( (z, t) = (1, 0) \).

The next result shows how the added structure of the kernel equation, when combined with the results of Chapter 2, can give general classification results for the generating functions of lattice path models.

**Theorem 21.** Let \( B(t) = \sum_{n \geq 0} b_n t^n \) be the generating function for the subclass of walks in the above model which return to the origin. Then \( B(t^p) \) is D-finite for some natural number \( p > 0 \).

**Proof.** Let \( p = - \min_{i \in S, j \in I} j \), so that \( z^p P(z) \) is a polynomial. If \( p \leq 0 \) then it is impossible for a walk to return to the origin, and the result holds trivially, so we assume \( p > 0 \). By the definition of \( C(z, t) \) we have

\[
b_n = \left[ z^0 \right][t^n] C(z, t) = \left[ z^0 \right][t^n] \frac{1}{1 - tP(z)}
= \left[ z^0 \right][t^p n] \frac{1}{1 - t^p z^p P(z)}
= \left[ z^p \right][t^n] \frac{1}{1 - t^p z^p P(z)}
= \left[ t^p \right] \Delta \left( \frac{1}{1 - t^p z^p P(z)} \right).
\]

Theorem 16 implies that the diagonal is some D-finite function \( R(t) = \sum_{n \geq 0} r_n t^n \), so that \( b_n = r_{np} \). Thus, if \( \omega = e^{2\pi i / p} \) we see

\[
B(t^p) = \sum_{n \geq 0} r_{np} t^{np} = \frac{1}{p} \left( R(t) + R(t \omega) + \cdots + R(t \omega^{p-1}) \right)
\]
is D-finite.

In the one dimensional case, Theorem 14 implies that \( B(t^p) \), as the diagonal of a bivariate rational function, is algebraic. Theorem 1 of Banderier and Flajolet [3] gives the explicit representation

\[
B(t) = t \sum_{j=1}^{p} \frac{z_j'(t)}{z_j(t)},
\]

where \( z_1(t), \ldots, z_p(t) \) are the (complete set of) \( p \) algebraic roots of \( 1 - tP(z) = 0 \) which approach zero as \( t \) approaches zero.

### 3.2 Lattice Walks in a Halfspace

We now consider walks restricted to a halfspace, following the presentation and solution given by Banderier and Flajolet [3]. Given a fixed set \( S \subseteq \mathbb{Z}^{d+1} \), we let \( \mathcal{C} \) denote the lattice path model taking steps in \( S \) whose walks have restricting region \( \mathcal{R} = \mathbb{Z}^{d} \times \mathbb{N} \). In order to argue combinatorially while making sure that we only consider walks in the halfspace \( \mathcal{R} \), we again use the *characteristic polynomial* \( P(x) \), now defined by

\[
P(x) = \sum_{(i_1, \ldots, i_{d+1}) \in S} x^{i_{d+1}} \in \mathbb{N}[x, x].
\]

For the following analytical arguments we assume that the greatest common divisor of the integer powers appearing in \( P(x) \) is one (if not, we can scale the \((d+1)\)st coordinate of \( \mathbb{Z}^{d+1} \) by this common divisor to obtain an isomorphic class with this property).

Let \(-a\) be the minimum of the exponents appearing in \( P(x) \) and \( b \) be the maximum of the exponents appearing in \( P(x) \). If \( a \leq 0 \) we are in the unrestricted case of the previous section, and if \( b \leq 0 \) then either there are no valid walks of non-zero length, or the model stays on the hyperplane \( z_{d+1} = 0 \) and is isomorphic to an unrestricted walk on a lower dimensional lattice. Thus, we assume \( a, b > 0 \) and for convenience write

\[
P(x) = \sum_{j=-a}^{b} p_j x^j, \quad \text{for } p_j \in \mathbb{N}.
\]

The (bi-variate) generating function \( C(x, t) = \sum_{i,n \geq 0} c_{i,n} x^i t^n \) where \( c_{i,n} \) denotes the number of walks of length \( n \) ending with \( z_{d+1} \) coordinate \( i \) is analytic for \(|x| \leq 1 \) and \(|z| < 1/|S|\), as
the number of walks of length $n$ in $\mathcal{C}$ is bounded by the $|\mathcal{S}|^n$ obtained in the unrestricted case. Define $C_n(x)$ and $D_k(t)$ by

$$C(x,t) = \sum_{n \geq 0} C_n(x)t^n = \sum_{k \geq 0} D_k(t)x^k.$$ 

As in the unrestricted case, applying the decomposition that a walk of length $n+1$ is a walk of length $n$ followed by a step in $\mathcal{S}$ gives a recurrence

$$C_0(x) = 1, \quad C_{n+1}(x) = P(x)C_n(x) - [x^<0]P(x)C_n(x) \quad \text{for } n \geq 0,$$

(3.3)

where $[x^<0]P(x)C_n(x)$ denotes the terms in the Laurent series $P(x)C_n(x)$ with negative exponents (subtracted out to keep the walks in the positive halfspace). This gives the kernel equation

$$(1 - tP(x))C(x,t) = 1 - t[x^<0](P(x)C(x,t))$$

$$= 1 - t \sum_{k=0}^{a-1} r_k(x)D_k(t),$$

(3.4)

where

$$r_k(x) = [x^<0](P(x)x^k) = \sum_{j=-a}^{-k-1} p_jx^{j+k}.$$

As the halfspace kernel equation (3.4) is more complex than the unrestricted kernel equation (3.2), deeper arguments are needed here. A detailed analysis of the roots of the kernel, which satisfy $1 - tP(x) = 0$, carried out by Banderier and Flajolet [3] shows that there are precisely $a$ roots in $u$, denoted $x_1(t), \ldots, x_a(t)$, which are analytic at the origin and approach zero as $t$ approaches zero (there are an additional $b$ branches which approach infinity as $t$ approaches zero). These considerations determine $C(x,t)$.

**Theorem 22** (Theorem 2, Banderier and Flajolet [3]). The bivariate generating function $C(x,t)$ is algebraic, and has the representation

$$C(x,t) = \frac{\prod_{j=1}^{a}(x - x_j(t))}{x^a(1 - tP(x))}.$$

In particular, the generating function for the total number of walks of length $n$ in $\mathcal{C}$ is the algebraic function

$$C(t) := C(1,t) = \frac{1}{1 - t|\mathcal{S}|} \prod_{j=1}^{a}(1 - x_j(t)),$$
and the generating function for the number of walks of length $n$ in $C$ which end on the hyperplane $z_{d+1} = 0$ is the algebraic function

$$B(t) := C(0, t) = \frac{(-1)^{a-1}}{p_{-a} t} \prod_{j=1}^{a} x_j(t).$$

These representations, combined with results in analytic combinatorics, also yield asymptotic information.

**Example 23** (Free Dyck paths). Let $S = \{-1, 1\}$ and consider walks which stay in the halfspace $\mathcal{R} = \mathbb{N}$. Here we have $P(x) = x^{-1} + x$, and the kernel equation (3.4) becomes

$$(1 - t(\overline{x} + x))C(x, t) = 1 - t[x^{<0}](\overline{(\overline{x} + x)C(x, t)}).$$

The only terms with negative exponents in the product $(\overline{x} + x)C(x, t)$ are those corresponding to terms of $C(x, t)$ which do not contain $x$. These are precisely the terms given by $C(0, t)$, which implies

$$(1 - t(\overline{x} + x))C(x, t) = 1 - t\overline{x}C(0, t),$$

or equivalently

$$(x - t(1 + x^2))C(x, t) = x - tC(0, t). \quad (3.5)$$

Solving $x - t(1 + x^2) = 0$ for $x$ gives two branches

$$x_1(t) = \frac{1 - \sqrt{1 - 4t^2}}{2t}, \quad x_2(t) = \frac{1 + \sqrt{1 - 4t^2}}{2t},$$

of which $x_1(t)$ is the branch which approaches zero as $t$ approaches zero. Substitution into Equation (3.5) gives

$$C(0, t) = \frac{1 - \sqrt{1 - 4t^2}}{2t^2},$$

so

$$C(x, t) = \frac{u - \frac{1 - \sqrt{1 - 4t^2}}{2t}}{x - t(1 + x^2)} = \frac{1 - 2xt - \sqrt{1 - 4t^2}}{2t(t + tx^2 - x)}.$$

Since

$$C(t) = C(1, t) = \frac{1}{2t} \left( \frac{\sqrt{1 - 4t^2}}{1 - 2t} - 1 \right)$$

has the asymptotic expansion $\sqrt{2}(1 - 2t)^{-1/2} + O(1 - 2t)$ at its singularity $t = 1/2$ closest to the origin, the transfer methods of Flajolet and Sedgewick [34] automatically imply that the total number of walks restricted to the halfspace grows asymptotically as $\frac{\sqrt{2}}{\sqrt{\pi}} \cdot n^{-1/2} \cdot 2^n$. \(<
Example 24 (The ballot problem). Returning to the ballot problem as given in Example 2, we wish to find the generating function for the number of walks ending at the non-negative integer \(k\) for the class of free Dyck paths. This is given by

\[
[x^k]C(x, t) = [x^k]\frac{x - x_1(t)}{(-1)(x - x_1(t))(x - x_2(t))}
\]

\[
= [x^k]\frac{1}{t(x_2(t) - x)}
\]

\[
= \frac{x_2(t)^{-(k+1)}}{t}
\]

\[
= \frac{x_1(t)^{k+1}}{t}.
\]

As \(x_1(t) = t(1 + x_1(t)^2)\) by our definition of \(x_1(t)\) as a root of the kernel, the Lagrange Inversion Theorem implies that the number of ballot orders given \(n\) ballots is

\[
[t^n][x^k]C(x, t) = [t^{n+1}]x_1(t)^{k+1} = \frac{k + 1}{n + 1}\left[\binom{n}{n-k}(1 + w^2)^{n+1}\right] = \frac{k + 1}{n + 1}\left(\frac{n + 1}{n - k}\right),
\]

if \(n \equiv k \pmod{2}\), and 0 otherwise.

\[\text{\textcopyright} \]

3.3 Lattice Walks in the Quarter Plane

Lattice path models in a halfspace can be considered to have a one dimensional restriction, in that the walks are restricted only in one coordinate; a clear extension is to consider walks with restrictions in several coordinates concurrently. Lattice path models which have restricting region \(\mathcal{R} = \mathbb{N}^2 \subseteq \mathbb{Z}^2\) are called models of lattice walks in the quarter plane. As seen in the preceding section, it is possible to enumerate a walk in \(\mathbb{Z}^{d+2}\) with restricting region \(\mathbb{Z}^d \times \mathbb{N}^2\) by studying a model restricted to \(\mathbb{N}^2\) through a projection: we only need to keep track of where a walk goes in its final two coordinates as the first \(d\) are unrestricted. The problem is that this gives rise to step sets with multiple versions of the same step, and the results in the quarter plane are not sufficiently general to deal with such weighted models. This leads to us considering the region \(\mathcal{R}\) instead of the more general region \(\mathbb{Z}^d \times \mathbb{N}^2\).

Although lattice walks in a halfspace always have algebraic generating functions, it was shown by Bousquet-Mélou and Petkovšek [19] that there are models staying in the quarter plane whose generating functions are not even D-finite. For this reason, lattice walks in the quarter plane have been useful tools for investigating properties of D-finiteness and refining methods which prove or disprove D-finiteness for generating functions. Much work
in this area has focused on walks which take unit or short steps – that is, models where $\mathcal{S} \subseteq \{\pm 1, 0\}^2$. The restriction to short steps bounds the degree of the kernel to be at most two – allowing its roots to be determined explicitly – and causes the kernel equation to have the (relatively) simple form given in Equation (3.6). These models were originally studied via kernel techniques in a probabilistic context by Fayolle et al. [32], and the treatment below begins by following a now standard method of argument popularized for quarter plane walks by the combinatorial work of Bousquet-Mélou and Mishna [18]. We discuss the models with D-finite generating functions in this section, with Chapter 4 dedicated to models with non D-finite generating functions.

Many subsets of $\{\pm 1, 0\}^2$ lead to models which contain no non-empty walks, or models which are combinatorially isomorphic to models restricted only to a halfspace (for instance, the model defined by $\mathcal{S} = \{(-1, 0), (0, 1), (0, -1)\}$ can never take its first step and thus can be viewed as a free Dyck path along the positive $y$-axis, solved in Example 23). Thus, in the following we assume that any set of steps $\mathcal{S}$ contains some step with 1 in its first coordinate and some step (possibly the same) with 1 in its second coordinate.
3.3.1 The Kernel Equation and Group of a Walk

Analogous to the unrestricted and halfspace cases, given a step set $S \subseteq \{\pm 1, 0\}^2$ we use the generating function

$$C(x, y, t) := \sum_{i,j,n \geq 0} c_{i,j,n} x^i y^j t^n,$$

where $c_{i,j,n}$ counts the number of walks of length $n$ taking steps from $S$ which stay in the positive quadrant and end at the point $(i, j)$, along with the characteristic polynomial

$$P(x, y) := \sum_{(i,j) \in S} x^i y^j \in \mathbb{Q}[x, y],$$

Again we can use the recursive structure of a walk of length $n$ to get the kernel equation satisfied by $C(x, y, t)$. As $C(0, y, t)$ (respectively $C(x, 0, t)$) gives the generating function of walks ending on the $y$-axis (respectively $x$-axis), and $S$ is restricted to contain only unit steps, the kernel equation in the quarter plane becomes

$$xy(1 - tP(x, y))C(x, y, t) = xy - tyC(0, y, t) - txC(x, 0, t) + \epsilon tC(0, 0, t),$$

(3.6)

where $\epsilon = 1$ if $(-1, -1) \in S$ and $\epsilon = 0$ otherwise (this compensates for subtracting off walks ending at the origin twice in the other terms). The additional variable present in the kernel $K(x, y) = 1 - tP(x, y)$ complicates the analysis by forcing one to consider algebraic surfaces solving $K(x, y)$ instead of algebraic curves (as in the halfspace case).

This complication led Bousquet-Mélou [16], followed by Bousquet-Mélou and Mishna [18], to borrow the notion of the group of a model from the probabilistic studies of Fayolle et al. [32]. If we define the Laurent polynomials $A_j(y)$ and $B_j(x)$, $j \in \{-1, 0, 1\}$, by

$$P(x, y) = xA_1(y) + A_0(y) + \bar{x}A_{-1}(y) = yB_1(x) + B_0(x) + \bar{y}B_{-1}(x),$$
Figure 3.3: The 56 non-isomorphic models with infinite group $G$.

then the bi-rational transformations $\Psi$ and $\Phi$ of the plane defined by

$$\Psi : (x, y) \mapsto \left( x \frac{A_1^{-1}(y)}{A_1(y)}, y \right)$$

$$\Phi : (x, y) \mapsto \left( x, \frac{B_1^{-1}(x)}{B_1(x)} \right),$$

fix $P(x, y)$ and thus $K(x, y)$. The group $G$ of a model is defined to be the group of bi-rational transformations of the $(x, y)$–plane generated by the involutions $\Psi$ and $\Phi$. Bousquet-Mélou and Mishna showed that, up to isomorphism, there are only 79 distinct models with unit steps: 23 whose corresponding group is finite and 56 whose corresponding group is infinite (see Figures 3.2 and 3.3).

As both $\Psi$ and $\Phi$ are involutions, to prove that $G$ is finite it is sufficient to find a natural number $n$ such that composing the group element $\Psi \circ \Phi$ with itself $n$ times yields the identity, a feat easily accomplished in a computer algebra system (assuming such $n$ exists and is of reasonable size). To prove that a group is of infinite order one can find an explicit point $(x_0, y_0)$ in the plane whose image under $G$ has infinite size, or show that the mapping $\Psi \circ \Phi$ never composes to the identity by analyzing its Jacobian at fixed points (Bousquet-Mélou and Mishna do both for the various cases in Figure 3.3).
### 3.3.2 Diagonals of Rational Functions

For convenience, given $g \in G$ and Laurent polynomial $A(x, y)$ we set $g(A(x, y)) := A(g(x, y))$. Furthermore, if $G$ has size $2n$ then any element $g \in G$ can be written uniquely as either $g = \Psi \circ \Phi \circ \cdots \circ \Phi$ or $g = \Psi \circ \Phi \circ \cdots \circ \Psi$, where there are $r < 2n$ terms in the composition, and we define $\text{sgn}(g) := (-1)^r$. In addition to determining whether the group of each model is finite, Bousquet-Mélou and Mishna also proved that 22 of the 23 models with finite group admit D-finite generating functions. Central to their argument is the following result.

**Proposition 25** (Bousquet-Mélou and Mishna [18], Proposition 5). Assume that the group $G$ is finite. Then

$$
\sum_{g \in G} \text{sgn}(g) g(xyC(x, y, t)) = \frac{1}{K(x, y, t)} \sum_{g \in G} \text{sgn}(g) g(xy).
$$

**Proof.** Define $x'$ and $y'$ by $(x', y) = \Psi(x, y)$ and $(x, y') = \Phi(x, y)$. Applying the maps $\Psi$ and $\Phi$ successively to the kernel equation gives

(id) \hspace{1cm} xyK(x, y)C(x, y, t) = xy - tyC(0, y, t) - txC(x, 0, t) + \varepsilon \cdot tC(0, 0, t)

($\Psi$) \hspace{1cm} x'yK(x, y)C(x', y, t) = x'y - tyC(0, y, t) - tx'C(x', 0, t) + \varepsilon \cdot tC(0, 0, t)

($\Phi\Psi$) \hspace{1cm} x'y'K(x, y)C(x', y', t) = x'y' - ty'C(0, y', t) - tx'C(x', 0, t) + \varepsilon \cdot tC(0, 0, t),

as both $\Psi$ and $\Phi$ fix $K(x, y)$. Note that $-tyC(0, y, t)$ and $-tx'C(x', 0, t)$ both appear on the right hand sides of successive equations, so taking an alternating sum of these three equations would cancel these terms. In fact, since $\Psi$ and $\Phi$ each fix one variable, continuing to compose the group generators in this manner and taking an alternating series of the resulting equations would cancel each unknown function of the form $C(0, Y, t)$ or $C(X, 0, t)$ arising on the right hand side. This follows from the finiteness of the group, as the compositions of group elements eventually return to the identity. The $\varepsilon tC(0, 0, t)$ term is also canceled as the group has even order, and the resulting equation gives the theorem. \hfill \Box

The procedure described in the proof of Theorem 25 is known in the literature as the **orbit sum method**, as one sums the kernel equation over orbits of the group generators, and it is crucial to the results in Chapter 5. In order to connect this to the D-finiteness of generating functions, we consider another operation on formal Laurent series. Given $F(x, y, t) \in F[x, y, y][[t]]$, we define $[x^{g_0}]y^{g_0}F(x, y, t)$ to be the formal power series obtained by taking only the non-negative powers of $x$ and $y$ in the series defining $F(x, y, t)$. 

Bousquet-Mélou and Mishna [18] show that \( [x^{\geq 0}] [y^{\geq 0}] F(x, y, t) \) is D-finite when \( F(x, y, t) \) is rational by relating this operator to the diagonal (we treat a more general case in Proposition 55 of Chapter 5). By examining Equation (3.7) for each of the 23 cases with finite group one gets the following.

**Theorem 26.** Let \( S \) be one of the 19 step sets with finite group which is not listed in Figure 3.4. Then \( C(x, y, t) = [x^{\geq 0}] [y^{\geq 0}] R(x, y, t) \), where \( R(x, y, t) \) is the rational function

\[
R(x, y, t) = \frac{1}{K(x, y, t) \sum_{g \in G} \text{sgn}(g) g(xy)},
\]

and is thus D-finite.

In particular, \( C(x, y, t) \) and \( C(t) = C(1, 1, t) \) – the generating function for the total number of walks in the quarter plane of length \( n \) – can be written as the diagonals of explicit rational functions. Chapter 5 is devoted to studying situations under which multidimensional complex analysis can be used to provide asymptotics and other information about lattice walks in the quarter plane, and their generalization to lattice walks in a positive orthant.

The four walks in (3.4) have both sides of their associated orbit sum equation (3.7) identically zero due to an element of the group fixing the product \( xy \) while having negative sign. Bousquet-Mélou and Mishna proved that the first three walks in (3.4) are algebraic (and thus can be written as diagonals of rational functions) by taking a modified ‘half-orbit sum’ and performing a detailed analysis, but the final model – known as *Gessel’s walk* – was classified by a computational approach outlined below. Bousquet-Mélou and Mishna conjectured that all 56 walks with an infinite group had non D-finite univariate generating functions, as previous work of Mishna [51] had also, but did not prove this for any model.

### 3.3.3 A Computer Algebra Approach

A computational approach to the asymptotics and classification of walks in the quarter plane by Bostan and Kauers [10, 11] relies on the fact that one can compute finite truncations of
the generating functions involved and use these truncations to guess algebraic or differential
equations which they satisfy. Given a set of steps \( S \), the recursive definition of a walk of
length \( n \) as a walk of length \( n - 1 \) followed by a single step corresponds to the recurrence
relation
\[
c_{i,j,n} = \sum_{(a,b) \in S} \epsilon_{i-a,j-b} c_{i-a,j-b,n-1}, \quad \epsilon_{i-a,j-b} = \begin{cases} 0 & : i - a < 0 \text{ or } j - b < 0 \\ 1 & : \text{otherwise} \end{cases}
\]
with the initial conditions \( c_{0,0,0} = 1 \) and \( c_{i,j,0} = 0 \) otherwise.

**Theorem 27.** Given step set \( S \subseteq \{\pm 1, 0\}^2 \), the first \( N \) terms of the generating function
\( C(t) = \sum_{n \geq 0} c_n t^n \) counting the number of walks restricted to the quarter plane taking steps
in \( S \) can be calculated in \( O(N^4) \) binary operations using \( O(N^2) \) bit space.

**Proof.** As \( S \) takes only unit steps, it must be that \( c_{i,j,n} = 0 \) if \( i > n \) or \( j > n \). Thus,
using the recurrence (3.8) to determine \( c_{i,j,n} \) from all non-zero terms of the form \( c_{i',j',n-1} \)
takes \( O(n^2) \) integer operations, and determining all terms \( c_{i,j,n} \) with \( 0 \leq i,j,n \leq N \) takes
\( O(1^2 + 2^2 + \cdots + N^2) = O(N^3) \) integer operations. Since there are \( |S|^n \) unrestricted walks
of length \( n \), we see \( c_{i,j,n} \leq |S|^n \) for all \( i,j \in \mathbb{N} \). The bit size of the integers at each step of
the recurrence grows at most linearly, so \( O(N^4) \) binary operations are sufficient to calculate
the truncation.

Unrolling the recurrence naively as discussed above would take \( O(N^4) \) bit space, however
once the terms of the form \( c_{i,j,n} \) have been calculated there is no need to store terms of the
form \( c_{i,j,n-1} \) – only their sum \( c_{n-1} = \sum_{0 \leq i,j \leq n} c_{i,j,n-1} \) is needed. This observation lowers the
binary space complexity to \( O(N^3) \), and the final result of \( O(N^2) \) is achieved by finding the
first \( N \) terms of \( C(t) \) in the modular field \( \mathbb{Z}/p_k\mathbb{Z} \) for \( O(N) \) machine primes \( p_1, \ldots, p_N \left[ \log |S| \right] \)
followed by a recovery of \( C(t) \) over the integers using the Chinese Remainder Theorem. \( \square \)

**Padé-Hermite Approximants**

Given a vector of formal power series
\[
F = (F_1, \ldots, F_r) \in K[[t]]^r
\]
and a vector of natural numbers
\[
d = (d_1, \ldots, d_r) \in \mathbb{N}^r,
\]
a vector $\mathbf{P} = (P_1, \ldots, P_r) \neq 0$ of polynomials in $K[t]$ is called a *Padé-Hermite approximant of type $\mathbf{d}$* for $F$ if

1. $\mathbf{P} \cdot F = O(t^{\sigma})$ – i.e., the lowest non-zero term in the dot product has exponent at least $\sigma$ – where $\sigma = \sum_{i=0}^{r} (d_i + 1) - 1$;

2. the degree of $P_i$ is at most $d_i$ for all $1 \leq i \leq n$.

**Example 28.** Given a power series $F(t) \in \mathbb{Q}[[t]]$ and $d \in \mathbb{N}$, if one takes $F_k = F(t)^{k-1}$ and $d = (d, \ldots, d)$ in the above definition then $\mathbf{P}$ is a Padé-Hermite approximant of type $\mathbf{d}$ if and only if $F(t)$ satisfies an algebraic equation of order $r$ with coefficients of degree at most $d$, up to order $t^{rd+r-1}$:

$$P_r(t) F(t)^{r-1} + \cdots + P_2(t) F(t) + P_1(t) = 0 \pmod{t^{rd+r-1}}.$$

$\triangleright$

**Example 29.** Given a power series $F(t) \in \mathbb{Q}[[t]]$ and $d \in \mathbb{N}$, if one takes $F_k = \frac{d^k}{d t^k} F(t)$ and $d = (d, \ldots, d)$ in the above definition then $\mathbf{P}$ is a Padé-Hermite approximant of type $\mathbf{d}$ if and only if $F(t)$ satisfies an linear differential equation of order $r$ with coefficients of degree at most $d$, up to order $t^{rd+r-1}$:

$$P_r(t) \frac{d^r}{d t^r} F(t) + \cdots + P_2(t) \frac{d^2}{d t^2} F(t) + P_1(t) \frac{d}{d t} C(t) = 0 \pmod{t^{rd+r-1}}.$$

$\triangleright$

These examples show how Padé-Hermite approximants allow one to guess an annihilating algebraic or differential equation for a truncation of a generating function. By construction, they always exist.

**Proposition 30.** *Every vector $\mathbf{F}$ admits a Padé-Hermite approximant of type $\mathbf{d} = (d_1, \ldots, d_r)$.***

**Proof.** By writing $P_i = \sum_{j=0}^{d_i} p_{i,j} t^j$, the condition $\mathbf{P} \cdot F = O(t^{\sigma})$ gives a homogeneous linear system with $\sigma = \sum_{i=1}^{r} (d_i + 1) - 1$ equations and $\sigma + 1$ unknowns $p_{i,j}$. $\square$

Furthermore, there is an efficient algorithm to calculate such an approximant.

**Theorem 31** (Beckermann and Labahn [5]). *Given the vector $\mathbf{F}$ it is possible to calculate a Padé-Hermite approximant of type $\mathbf{d}$ in $O(MM(r, \sigma) \log \sigma)$ operations over the base field*
$K$, where $MM(r, \sigma)$ is the number of operations required to multiply two $r \times r$ matrices whose entries are polynomials of degree at most $\sigma$ modulo $t^{\sigma+1}$. The Coppersmith–Winograd algorithm for matrix multiplication implies $O(\log(\sigma)) = O(r^{2.376} \log(\sigma))$, where $M(\sigma)$ is the cost of multiplying two degree $\sigma$ polynomials over the field $K$. If $K = \mathbb{Q}$ and $d = (d, d, \ldots, d)$ then $O(\log(\sigma)) = O(r^{3.376} d \log^2(rd) \log \log(rd))$.

This allows us to conjecture whether or not the generating function $C(t)$ is algebraic or D-finite: pick some bounds $r$ and $d$ on the possible order and degree of an annihilating polynomial / linear differential equation, generate a sufficient number $N$ of terms of $C(t)$ using Theorem 27, then calculate the corresponding Padé-Hermite approximant using Theorem 31. A guessed algebraic or differential equation gives a linear recurrence with constant coefficients that the coefficients $c_n$ satisfy, and one may gain confidence that $C(t)$ truly satisfies the guessed equation by checking that the coefficients of $C(t)$ beyond those used to calculate the approximant (for instance up to $n = 2N$) satisfy the calculated recurrence. If the recurrence does not hold for additional terms, one can increase $r$ and $d$ and repeat the procedure (or admit that $C(t)$ may be non D-finite if $r$ and $d$ are sufficiently large). The guessing process has been implemented in the Maple package gfun (see Salvy and Zimmermann [58]).

Applications to Lattice Path Models

A formal power series $F(t) \in \mathbb{Q}[[t]]$ is called a $G$-function if it is D-finite, its radius of convergence in $\mathbb{C}[[t]]$ is positive, and there exists a constant $K > 0$ such that the common denominator of $a_0, \ldots, a_n$ is bounded by $K^n$ for all $n \geq 0$. As $c_n \leq |S|^n$ in each model, the generating function $C(t)$ represents the Taylor series of an analytic function near the origin with natural number coefficients, so if $C(t)$ is D-finite then it is a G-function. Bostan and Kauers [10] used additional algebraic properties of the minimal differential operator annihilating a G-series to give a “fast algorithmic filter” for testing the legitimacy of guessed equations. They were able to guess algebraic and/or differential equations for each of the 23 models with finite group in Figure 3.2, and also conjectured that the 56 models with infinite group in Figure 3.3 were non D-finite.

A result of Garoufalidis [36] shows that the dominant asymptotics of $c_n$ for any G-function $C(t)$ must have the form $c_n \sim \kappa \rho^n n^\alpha (\log n)^\beta$ where $\kappa \in \mathbb{R}, \rho \in \overline{\mathbb{Q}}, \alpha \in \mathbb{Q}$, and $\beta \in \mathbb{N}$. Furthermore, given the differential equation satisfied by D-finite $C(t)$ it is possible
CHAPTER 3. LATTICE PATH MODELS

S D-fin. Alg. Asymptotics

3,4 — \( \frac{4}{\pi} \cdot \frac{4^n}{n} \)

3,5 — \( \frac{2}{\pi} \cdot \frac{4^n}{n} \)

3,8 — \( \frac{\sqrt{6}}{\pi} \cdot \frac{6^n}{n} \)

3,6 — \( \frac{8}{3\pi} \cdot \frac{8^n}{n} \)

5,16 — \( \frac{\sqrt{3}}{2\sqrt{2}} \cdot \frac{3^n}{\sqrt{n}} \)

5,20 — \( \frac{\sqrt{5}}{2\sqrt{2\pi}} \cdot \frac{5^n}{\sqrt{n}} \)

5,15 — \( \frac{4}{3\sqrt{\pi}} \cdot \frac{4^n}{\sqrt{n}} \)

5,18 — \( \frac{2\sqrt{3}}{3\sqrt{\pi}} \cdot \frac{6^n}{\sqrt{n}} \)

2,3 2,2 \( \frac{3\sqrt{7}}{2\sqrt{\pi}} \cdot \frac{3^n}{n^{3/2}} \)

2,3 2,2 \( \frac{3\sqrt{3}}{2\sqrt{\pi}} \cdot \frac{6^n}{n^{3/2}} \)

4,9 6,8 \( \frac{2\sqrt{2}}{\Gamma(1/4)} \cdot \frac{3^n}{n^{3/4}} \)

3,5 8,9 \( \frac{4\sqrt{3}}{3\Gamma(1/3)} \cdot \frac{4^n}{n^{3/4}} \)

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>3,4</td>
<td>—</td>
<td>( \frac{4}{\pi} \cdot \frac{4^n}{n} )</td>
<td>5,24</td>
<td>—</td>
<td>( \frac{\sqrt{5}}{3\sqrt{2\pi}} \cdot \frac{5^n}{\sqrt{n}} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3,5</td>
<td>—</td>
<td>( \frac{2}{\pi} \cdot \frac{4^n}{n} )</td>
<td>5,24</td>
<td>—</td>
<td>( \frac{\sqrt{7}}{3\sqrt{3\pi}} \cdot \frac{7^n}{\sqrt{n}} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3,8</td>
<td>—</td>
<td>( \frac{\sqrt{6}}{\pi} \cdot \frac{6^n}{n} )</td>
<td>4,15</td>
<td>—</td>
<td>( \frac{12\sqrt{3}}{\pi} \cdot \frac{(2\sqrt{3})^n}{n^2} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3,6</td>
<td>—</td>
<td>( \frac{8}{3\pi} \cdot \frac{8^n}{n} )</td>
<td>5,18</td>
<td>—</td>
<td>( \frac{\sqrt{3}(1+\sqrt{3})^{7/2}}{2\pi} \cdot \frac{(2+2\sqrt{3})^n}{n^2} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5,16</td>
<td>—</td>
<td>( \frac{\sqrt{3}}{2\sqrt{2\pi}} \cdot \frac{3^n}{\sqrt{n}} )</td>
<td>5,24</td>
<td>—</td>
<td>( \frac{12\sqrt{3}}{\pi} \cdot \frac{(2\sqrt{3})^n}{n^2} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5,20</td>
<td>—</td>
<td>( \frac{\sqrt{5}}{2\sqrt{2\pi}} \cdot \frac{5^n}{\sqrt{n}} )</td>
<td>5,24</td>
<td>—</td>
<td>( \frac{\sqrt{6(379+156\sqrt{3})^{1/2}}}{5\sqrt{95\pi}} \cdot \frac{(2+2\sqrt{6})^n}{n^2} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5,15</td>
<td>—</td>
<td>( \frac{4}{3\sqrt{\pi}} \cdot \frac{4^n}{\sqrt{n}} )</td>
<td>4,16</td>
<td>—</td>
<td>( \frac{24\sqrt{3}}{\pi} \cdot \frac{(2\sqrt{2})^n}{n^2} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5,18</td>
<td>—</td>
<td>( \frac{2\sqrt{3}}{3\sqrt{\pi}} \cdot \frac{6^n}{\sqrt{n}} )</td>
<td>5,19</td>
<td>—</td>
<td>( \frac{\sqrt{3}(1+\sqrt{2})^{7/2}}{\pi} \cdot \frac{(2+2\sqrt{2})^n}{n^2} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2,3 2,2</td>
<td>( \frac{3\sqrt{7}}{2\sqrt{\pi}} \cdot \frac{3^n}{n^{3/2}} )</td>
<td>4,12 6,8</td>
<td>( \frac{3\sqrt{7}}{\Gamma(1/4)} \cdot \frac{3^n}{n^{3/4}} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2,3 2,2</td>
<td>( \frac{3\sqrt{3}}{2\sqrt{\pi}} \cdot \frac{6^n}{n^{3/2}} )</td>
<td>2,3 4,4</td>
<td>( \frac{6\sqrt{3}}{\Gamma(1/4)} \cdot \frac{6^n}{n^{3/4}} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4,9 6,8</td>
<td>( \frac{2\sqrt{2}}{\Gamma(1/4)} \cdot \frac{3^n}{n^{3/4}} )</td>
<td>3,5</td>
<td>—</td>
<td>( \frac{8}{\pi} \cdot \frac{4^n}{n^2} )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3.1: Asymptotics guessed by Bostan and Kauers [10] for the finite group models.

(see Flajolet and Sedgewick [34]) to determine the corresponding values of \( \alpha, \rho, \) and \( \beta \). This allowed Bostan and Kauers to guess dominant asymptotics for each of the 23 walks with finite group – guesses which were later partially verified by Johnson et al. [41] and Fayolle and Raschel [31], among others (the constants \( \kappa \) were additionally guessed by Bostan and Kauers by computing high accuracy numerical truncations obtained from generating terms of the counting sequence). Their results are present in Table 3.1: the columns D-fin. and Alg. contain the degrees and orders of the minimal differential equations and polynomials annihilating the generating function \( C(t) \) for each model. For instance, Gessel’s walk – the final model in the table, with step set \( S = \{(-1,0), (1,0), (-1,-1), (1,1)\} \) – satisfies a differential equation of order 3 with polynomial coefficients of degree 5, and a degree 8 polynomial in the variable \( y \) whose coefficients are polynomials of degree at most 9 in \( t \).

Once one has these guessed an annihilating equation it is possible, but not automatic, to
prove rigorously that the generating function \( C(t) \) satisfies the guess. In a separate paper, Bostan and Kauers [11] were the first to prove that Gessel’s walk has a D-finite generating function (although Kauers et al. [43] had previously shown that the generating function for the number of walks using these steps which return to the origin is D-finite, also by a computational approach). In fact, they proved that the multivariate generating function \( C(x, y, t) \) is algebraic using this ‘guess-and-check’ method. Before this computational proof there was much debate between researchers on whether or not the generating function for Gessel’s walk was even D-finite, and Bostan and Kauers were surprised to see results implying algebraicity while originally attempting to guess annihilating differential equations.

We outline the argument in the simpler case of Kreweras Walks; incidentally, Kreweras [47] found a closed form for the number of walks in this model returning to the origin in 1965 by partially guessing (without this computational technology, of course) a solution to a recurrence which the counting sequence satisfies and then applying hypergeometric identities!

**Example 32** (Bostan and Kauers [11], Section 2). Consider the model of lattice walks in the quarter plane with step set \( S = \{(-1, 0), (0, -1), (1, 1)\} \), called Kreweras Walks. Here, the kernel equation (3.6) becomes

\[
(xy - t(x + y + x^2y^2))C(x, y, t) = xy - xtC(x, 0, t) - ytC(0, y, t).
\]

Solving \( xy - t(x + y + x^2y^2) \) by the quadratic formula gives the root

\[
Y_1(x, t) = \frac{x - t - \sqrt{x^2 - 2tx + t^2 - 4t^2x^3}}{2tx^2} = t + \frac{1}{x}t^2 + \frac{1 + x^3}{x^2}t^3 + O(t^4) \in \mathbb{Q}[x, t][[t]],
\]

and substitution of \( y = Y_1 \) in the kernel equation (which is valid as the constant term of \( Y_1 \) as a power series in \( t \) is zero) gives

\[
0 = xY_1(x, t) - xtC(x, 0, t) - Y_1(x, t)C(0, Y_1(x, t), t).
\]

Since the set of steps \( S \) is symmetric over the line \( y = x \), the definition of \( C(x, y, t) \) as the generating function marking the length and endpoint of a walk implies \( C(x, y, t) = C(y, x, t) \). Thus, \( C(x, 0, t) \) satisfies the functional equation

\[
U(x, t) = \frac{Y_1(x, t)}{t} - \frac{Y_1(x, t)}{x}U(Y_1(x, t), t).
\] (3.9)
In fact, if \( U(x,t) \) are \( U_2(x,t) \) are power series such that

\[
U_1(x,t) = \frac{Y_1(x,t)}{t} - \frac{Y_1(x,t)}{x} U_1(Y_1(x,t), t)
\]

and

\[
U_2(x,t) = \frac{Y_1(x,t)}{t} - \frac{Y_1(x,t)}{x} U_2(Y_1(x,t), t),
\]

then

\[
U_1(x,t) - U_2(x,t) = - \frac{Y_1(x,t)}{x} (U_1(Y_1(x,t), t) - U_2(Y_1(x,t), t)),
\]

which implies \( U_1(x,t) = U_2(x,t) \) as otherwise the lowest powers of \( t \) appearing on each side of the equation would not match. Thus, \( C(x,0,t) \) is the unique solution of (3.9).

Bostan and Kauers used Padé-Hermite approximants to guess that \( F(u,t) \) admits a rational parametrization (it is a curve of genus zero over \( \mathbb{Q}(x) \)), to prove that \( U(x,t) \) satisfies

\[
P(u,t) = (16x^3t^4 + 108t^4 - 72xt^3 + 8x^2t^2 - 2t + x)
\]

\[
+ (96x^2t^5 - 48x^3t^4 - 144t^4 + 104xt^3 - 16x^2t^2 + 2t - x)u
\]

\[
+ (48x^4t^6 + 192xt^6 - 264x^2t^5 + 64x^3t^4 + 32t^4 - 32xt^3 + 9x^2t^2)u^2
\]

\[
+ (192x^3t^7 + 128t^7 - 96x^4t^6 - 192xt^6 + 128x^2t^5 - 32x^3t^4)u^3
\]

\[
+ (48x^5t^8 + 192x^2t^8 - 192x^3t^7 + 56x^4t^6)u^4 + (96x^4t^9 - 48x^5t^8)u^5 + 16x^6t^{10}u^6,
\]

where \( P(u,t) \in \mathbb{Q}((x))[u,t] \). Since \( P(1,0) = 0 \) while \( P_u(1,0) = -x \), which is invertible in \( \mathbb{Q}((x)) \), the Implicit Function Theorem implies that there exists a unique root of \( P(u,t) \) in the ring \( \mathbb{Q}((x))[u,t] \) with constant term 1. Thus, there is at most one root of \( P(u,t) \) in \( \mathbb{Q}[[x,t]] \). Showing that such a root exists is harder, and Bostan and Kauers used the fact that \( P(u,t) \) admits a rational parametrization (it is a curve of genus zero over \( \mathbb{Q}(x) \)), to prove that there exists \( C_{cand}(x,t) \in \mathbb{Q}[[x,t]] \) such that \( P(C_{cand}(x,t), t) = 0 \).

Finally, we let

\[
F(x,t) = \frac{Y_1(x,t)}{t} - \frac{Y_1(x,t)}{x} C_{cand}(Y_1(x,t), t).
\]

As \( Y_1(x,t) \) is algebraic with annihilating polynomial \( xy - t(x + y + x^2y^2) \) and \( C_{cand}(x,y) \) is algebraic with annihilating polynomial \( P(u,t) \), one can use results in elimination theory (see Cox et al. [27]) to prove that \( F(x,t) \) is annihilated by \( P(u,t) \). But \( P(u,t) \) admits a unique power series root, so \( F(x,t) = C_{cand}(x,t) \) and \( C_{cand}(x,t) \) satisfies Equation (3.9)!

Since \( C(x,0,t) \) was the unique solution to Equation (3.9), \( C(x,0,t) \) is the algebraic function \( C_{cand}(x,t) \), and the kernel equation then implies that the full generating function \( C(x,y,t) \) is algebraic.

\[\square\]
Bostan and Kauers needed only the first 80 terms in $t$ to guess the polynomial $P(u, t)$ for the Kreweras walk generating function. As the step set for the Gessel walk model does not have symmetry across the line $x = y$, more detail must be taken and two annihilating polynomials must be guessed and then proven. Due to an increased complexity in the corresponding equations 1000 terms were used to make the guesses, and printed out the polynomials in question would fill approximately 30 pages! With the efficient method of calculating Padé-Hermite approximants given by Theorem 31, the bottleneck in the computational component of this method comes from generating the initial truncation of the generating functions. Indeed, when moving up to higher dimensions an increased order of magnitude in complexity to find counting sequences makes such guessing harder (though not impossible).
Chapter 4

Non D-Finite Models in the Quarter Plane

In this chapter we illustrate how to prove that the generating function of a lattice path model is non D-finite. This is an inherently more difficult problem than proving that a generating function is D-finite, as it asks one to prove an absence of structure instead of its existence.

Due to this difficulty, only five of the 56 models in Figure 3.3 with infinite group have a generating function $C(t)$ which has been proven non D-finite. The majority of this chapter shows how to use a kernel method variant called the iterated kernel method to prove non D-finiteness and give dominant asymptotics for the counting sequences of these models. Unfortunately, the iterated kernel method requires that the roots of the kernel in $x$ and $y$ have a power series expansion of a certain form, and consequently it cannot be extended to other 2D models with short steps in its present form. The final section of the chapter is dedicated to related developments which give further evidence that the remaining 51 models with an infinite group also admit non D-finite generating functions.

\begin{figure}[h]
\centering
\begin{align*}
\mathcal{A} = & \quad \mathcal{B} = \\
\mathcal{C} = & \quad \mathcal{D} = \\
\mathcal{E} = & \\
\end{align*}
\caption{Step sets of the five singular models}
\end{figure}
1. Determine the kernel equation of a model
2. Find power series root $Y_+(x, t)$ of the kernel
3. Write $C(x, 0, t)$ as an infinite series of iterates of $Y_+(x, t)$
4. Find a distinct singularity belonging to each iterate
5. Show $C(t)$ admits an infinite number of singularities

Figure 4.2: The iterated kernel method to prove non D-finiteness, in brief.

4.1 The Iterated Kernel Method

The iterated kernel method – an argument inspired by the work of Bousquet-Mélou and Petkovšek [19], and Janse van Rensburg et al. [40] – applies to the five models which have their step sets listed in Figure 4.1, known as the singular models. It was adapted by Mishna and Rechnitzer [52] to show that the generating functions $C(x, y, t)$ and $C(t)$ corresponding to the models $A$ and $D$ were non D-finite. The arguments given here come from Melczer and Mishna [50], which adapts the results of Mishna and Rechnitzer to prove that the remaining three singular models admit non D-finite counting generating functions $C(x, y, t)$ and $C(t)$.

The approach of the method is to give an explicit expression for the generating function in terms of an infinite series whose terms annihilate the kernel. In addition to proving non D-finiteness, these expressions give asymptotics for the number of walks in each model and more efficient algorithms for generating the number of walks (compared to Theorem 27). There are five main steps, listed in Figure 4.2.

4.1.1 Find the roots of the kernel.

Recall the kernel equation

$$xyK(x, y)C(x, y, t) = xy - tyC(0, y, t) - txC(x, 0, t), \quad (4.1)$$
where $K(x, y) = 1 - tP(x, y)$ and
\[
P(x, y) = xA_1(y) + A_0(y) + yA_{-1}(y) = yB_1(x) + B_0(x) + yB_{-1}(x).
\]

We are interested in the solutions to the kernel equation of the form
\[
K(x, Y_+(x; t)) = K(x, Y_-(x; t)) = K(X_+(y; t), y) = K(X_-(y; t), y) = 0,
\]
and these algebraic functions are easily determined since the kernel is a quadratic:
\[
Y_±(x, t) = \frac{(1 - tB_0(x)) \mp \sqrt{(B_0(x)^2 - 4B_1(x)B_{-1}(x)) t^2 - 2B_0(x)t + 1}}{2tB_1(x)}
\]
\[
X_±(y, t) = \frac{(1 - tA_0(y)) \mp \sqrt{(A_0(y)^2 - 4A_1(y)A_{-1}(y)) t^2 - 2A_0(y)t + 1}}{2tA_1(y)}.
\]

Each singular model contains the steps $(-1, 1)$ and $(1, -1)$, and at least one other step, which prevents degeneracy in the quadratic. We define the sequence of functions $Y_i(x, t)$ for $i \geq 0$ by the recurrence $Y_{n+1}(x, t) = Y_n(Y_+(x, t))$ with base case $Y_0(x, t) = x$, and the sequence $X_i(y, t)$ analogously. In addition, we define $Y_i(x, t)$ and $X_i(y, t)$ for $i < 0$ by the recurrence $Y_{-(n+1)}(x, t) = Y_{-n}(Y_-(x, t))$. Throughout this chapter we use $Y_n$ to refer only to terms having positive index and $Y_{-n}$ to refer to terms having negative index. Furthermore, as $t$ is a formal variable used in the generating function (and is never evaluated) we suppress $t$ and write $Y_n(x)$ to lighten notation. We express the generating functions in terms of the iterated compositions $Y_i(x)$ and $X_i(y)$, hence the name \textit{iterated kernel method}.

### 4.1.2 Determine an explicit expression for $C(x, 0, t)$.

Consider first the models $A, B, \text{ and } C$ (defined in Figure 4.1) whose step sets are symmetric with respect to the line $y = x$. This symmetry implies $C(x, y, t) = C(y, x, t)$ and $Y_i(x) = X_i(x)$ for all $i \in \mathbb{Z}$. Furthermore, substituting $x = Y_n(x)$ into the kernel relation $K(x, Y_+(x)) = 0$ implies that
\[
K(Y_n, Y_{n+1}) = K(Y_n(x), Y_+(Y_n(x))) = 0
\]
for all $n \in \mathbb{N}$. Thus, substituting into the kernel equation (4.1), we find for each $n$
\[
0 = Y_n(x)Y_{n+1}(x) + K(0, Y_{n+1}(x))C(0, Y_{n+1}(x), t) + K(0, Y_n(x))C(0, Y_n(x), t).
\]
CHAPTER 4. NON D-FINITE MODELS IN THE QUARTER PLANE

<table>
<thead>
<tr>
<th>Model</th>
<th>Asymptotic estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$\kappa_A \cdot 3^n + O\left( (2\sqrt{2})^n \right)$ $\kappa_A = 0.17317888\ldots$</td>
</tr>
<tr>
<td>$B$</td>
<td>$\kappa_B \cdot 4^n + O\left( (1 + 2\sqrt{2})^n \right)$ $\kappa_B = 0.15194581\ldots$</td>
</tr>
<tr>
<td>$C$</td>
<td>$\kappa_C \cdot 5^n + O\left( (1 + 2\sqrt{3})^n \right)$ $\kappa_C = 0.38220125\ldots$</td>
</tr>
</tbody>
</table>

Table 4.1: Estimates for $[t^n]C(t)$; a modern computer can find $\kappa_S$ to a thousand decimal places in seconds.

An alternating sum of these equations then yields an expression for $K(0, x)C(0, x, t)$:

$$0 = \sum_{n=0}^{\infty} (-1)^n \left( n Y_n Y_{n+1} + K(0, Y_{n+1})C(0, Y_{n+1}, t) + K(0, Y_n)C(0, Y_n, t) \right)$$

$$= K(0, x)C(0, x, t) + \sum_{n=0}^{\infty} (-1)^n Y_n(x)Y_{n+1}(x),$$

and the infinite series converges as a formal power series as $Y_1(x) = xt + xO(t^2)$ implies $Y_n(x) = O(t^n)$. Note that one cannot obtain an equivalent infinite series representation using the iterates $Y_{-n}(x)$ with negative index as they do not have an expansion of this form. Rearranging and evaluating at $x = 1$ gives an expression for the generating function of walks returning to the $y$-axis (which by symmetry is the same as those returning to the $x$-axis):

$$C(0, 1, t) = \frac{1}{t} \sum_{n=0}^{\infty} (-1)^n Y_n(1)Y_{n+1}(1).$$  \(4.5\)

Furthermore, substituting $x = 1$ and $y = 1$ into the kernel equation (4.1) gives the generating function of walks ending anywhere in the quarter plane:

$$C(1, 1, t) = \frac{1 - 2tC(0, 1, t)}{1 - t|S|} = \frac{1}{1 - t|S|} \left( 1 - 2 \sum_{n=0}^{\infty} (-1)^n Y_n(1)Y_{n+1}(1) \right).$$  \(4.6\)

In fact, one can find the functions $Y_n(x)$ explicitly – as we do in Theorem 34 below – which allows for the calculation of the dominant asymptotics of the symmetric models.

**Theorem 33** (Melczer and Mishna [50], Theorem 1). For each of the symmetric models $S \in \{A, B, C\}$, the number of walks grows asymptotically as

$$[t^n]C(t) = \kappa_S \left( \frac{1}{|S|} \right)^n + O \left( (a_0 + 2\sqrt{a_1a_{-1}})^n \right),$$  \(4.7\)

where $a_i = A_i(1)$ and $\kappa_S$ is a constant which can be calculated to arbitrary precision using Equation (4.5).
Proof. As \( C(t) = \frac{1 - 2tC(0, 1, t)}{1 - |S|t} \), we show that \( C(t) \) admits a simple pole at \( |S|^{-1} \), and that this is the dominant singularity (the singularity closest to the origin, which determines dominant asymptotics). The asymptotic expression in Equation (4.7) is then a consequence of evaluating the residue at this value, and bounding the dominant singularity of \( C(0, 1, t) \).

To accomplish this, we first consider the class \( \mathcal{H} \) of walks with steps from \( S \) restricted to the halfspace \( \mathcal{R} = \mathbb{Z} \times \mathbb{N} \) which end on the \( y \)-axis. The methods discussed in Section 3.2 gives the generating function of \( \mathcal{H} \) as:

\[
H(t) = \frac{(1 - ta_0) - \sqrt{(1 - ta_0)^2 - 4t^2a_{-1}a_1}}{2t^2a_1a_{-1}}.
\]

In particular, the dominant singularity of \( H(t) \) is \( t = \frac{1}{a_0 + 2\sqrt{a_1a_{-1}}} \). Now, the set of quarter plane walks with steps from \( S \) which return to the \( x \)-axis is a subset of this set, and so

\[
[t^n]C(0, 1, t) \leq [t^n]H(t).
\]

Consequently, as the exponential growth of a sequence is given by the reciprocal of its generating function’s dominant singularity, \( C(0, 1, t) \) is convergent for \( 0 \leq t \leq a_0 + 2\sqrt{a_1a_{-1}} < |S| \), where the latter inequality holds as \( a_{-1} = 1 \), \( a_0 \in \{0, 1\} \) and \( a_1 \in \{1, 2, 3\} \) in each case. Thus, the singularity of \( C(t) \) at \( |S|^{-1} \) is indeed dominant.

We also need to verify that \( C(0, 1, |S|^{-1}) \neq |S|/2 \) to show that \( |S|^{-1} \) is not a removable singularity. Using the explicit expression for \( 1/Y_n \) given in Theorem 34 one can show that \( Y_n(1)Y_{n+1}(1) \) is monotonically decreasing when \( t = 1/|S| \), so that the error on the \( N \)th partial sum of the alternating series is bounded by \( Y_{N+1}(1)Y_{N+2}(1) \). Numerically evaluating the \( 10 \)th partial sum is sufficient to bound \( 1 - 2\sum_{n}(-1)^nY_n(1)Y_{n+1}(1) \) away from 0 in each case. The results are summarized in Table 4.1.

We have shown that the dominant singularity of \( C(t) \) is the simple pole at \( t = |S|^{-1} \). The residue is \( C(0, 1, |S|^{-1}) \) which, when evaluated with suitable precision, gives the stated constants in Table 4.1. The sub-dominant factor comes from the fact that the dominant singularity of \( H(t) \) bounds the modulus of the dominant singularity of \( C(0, 1, t) \).

\[ \square \]

4.1.3 Exhibit distinct singularities of \( Y_n \).

(a) Locate the singularities

In order to simplify the argument for non D-finiteness, we re-parametrize by substituting \( t = \frac{q}{1+q^2} \) in the kernel equation. This construction then gives power series \( Y_n(x, q) \) and
Model | Recurrence | $Y_n(q)$
---|---|---
$A$ | $Y_n(q) = (q + \frac{1}{q})Y_{n-1}(q) - Y_{n-2}(q)$ | $(q^2-q^{2n})+q(q^{2n}-1)Y_{n}(q)\over q^n(q^2-1)$
$B, C$ | $Y_n(q) = (q + \frac{1}{q})Y_{n-1}(q) - Y_{n-2}(q) - 1$ | $q(q-1)(q^{2n}-1)Y_{1}(q) + (q+q^n)(2q^{n+1}-q^n+q^2-2q)\over q^n(q+1)(q-1)^2$

Table 4.2: The recurrences and solutions for models $A, B,$ and $C$.

$\mathcal{Y}_n(x, q)$ in $\mathbb{Q}[x, q]$ for $n \in \mathbb{N}$ such that

$$C \left( 1, 0, \frac{q}{1+q^2} \right) = (q + 1/q) \sum_{n=0}^{\infty} (-1)^n \mathcal{Y}_n(1, q) \mathcal{Y}_{n+1}(1, q).$$

(4.8)

To lighten notation, we let $\mathcal{Y}_k(q) := \mathcal{Y}_k(1, q)$ and $\mathcal{Y}_k(q) := 1\over \mathcal{Y}_k(1, q)$. The following theorem gives $\mathcal{Y}_n$ explicitly.

**Theorem 34.** For each of the models $A, B,$ and $C$, the functions $\mathcal{Y}_n(q)$ satisfy the linear recurrence relations shown in Table 4.2, and have the explicit form listed there.

**Proof.** For each model, expanding the polynomial

$$p(z) = \prod_{(j,k) \in \{\pm 1\}^2} (z - \mathcal{Y}_j(\mathcal{Y}_k(z, q), q)),$$

implies $p(z) = (x-z)^2r(z)$, where $r(z) \in \mathbb{R}(x,t)[z]$ is non-zero at $z = x$. Thus, two roots of $p(z)$ are equal to $x$, and examination of the initial terms of a Taylor series in $t$ shows that

$$\mathcal{Y}_-(\mathcal{Y}_+(x, q), q) = \mathcal{Y}_+(\mathcal{Y}_-(x, q), q) = x$$

for each of the symmetric models. The recurrence then follows from noting that the sum and product of $\mathcal{Y}_-(x, q)$ and $\mathcal{Y}_+(x, q)$ can be expressed in terms of the coefficients of $K(x, y)$, and the explicit form follows from solving the linear recurrence.

In fact, this explicit form allows us to prove that the formal power series representation (4.8) gives a representation of $C \left( 1, 0, \frac{q}{1+q^2} \right)$ as the Taylor series of an analytic function at the origin.

**Proposition 35.** For the models $A, B,$ and $C$, the sum $(q + 1/q) \sum_{n=0}^{\infty} (-1)^n \mathcal{Y}_n(q) \mathcal{Y}_{n+1}(q)$ is convergent for all $q \in \mathbb{C}$ with $|q| \neq 1$, except possibly at the set of points defined by the singularities of the $\mathcal{Y}_n(q)$.
CHAPTER 4. NON D-FINITE MODELS IN THE QUARTER PLANE

Model \( \sigma_n(q) \)

\[
\begin{align*}
A & : \quad \alpha_n(q) = q^{4n} + q^{2n+2} - 4q^{2n} + q^{2n-2} + 1 \\
B & : \quad \beta_n(q) = (q^{2n-1} + (q^3 - 2q^2 - 2q + 1)q^{n-2} + 1) (q^{2n+1} + (q^3 - 2q^2 - 2q + 1)q^{n-1} + 1) \\
C & : \quad \gamma_n(q) = q^2(1 + q^2 - q)(1 + q^{4n}) + q(q^2 - 3q + 1)(q + 1)^2(q^n + q^{2n}) \\
& \quad \quad \quad + q^{2n}(1 - q^2 - 4q + 14q^3 - 4q^5 - q^4 + q^6)
\end{align*}
\]

Table 4.3: The singularities of \( Y_n(q) \) satisfy the polynomial equation \( \sigma_n(q) = 0 \).

**Proof.** The result follows from the ratio test applied to the explicit formulas for \( Y_n(q) \).

When \( |q| < 1 \):

\[
\lim_{n \to \infty} \left| \frac{Y_{n+1}(q)Y_{n+2}(q)}{Y_n(q)Y_{n+1}(q)} \right| = \lim_{n \to \infty} \left| \frac{Y_n(q)}{Y_{n+2}(q)} \right| = |q|^2 < 1,
\]

and when \( |q| > 1 \):

\[
\lim_{n \to \infty} \left| \frac{Y_{n+1}(q)Y_{n+2}(q)}{Y_n(q)Y_{n+1}(q)} \right| = \lim_{n \to \infty} \left| \frac{Y_n(q)}{Y_{n+2}(q)} \right| = \frac{1}{|q|^2} < 1.
\]

By Theorem 13, to prove that \( C(1,1,t) \) is non D-finite it is sufficient to prove that \( C \left( 1,1,\frac{q}{1+q^2} \right) \) has an infinite number of singularities.

In order to argue about singularities, we find a family of polynomials \( \sigma_n(q) \) that the roots of \( Y_n(q) \) satisfy: the polynomials in Table 4.3 are obtained by manipulating the explicit expressions given above. Unfortunately, extraneous roots are introduced during the algebraic manipulations when an equation is squared to remove the square root present. In fact, the extraneous roots are exactly those which correspond to a negative sign in front of the square root. As \( Y_{-n}(q) \) satisfies the same recurrence relation as \( Y_n(q) \), and has the same initial condition up to a reversal of a square root sign, the set of roots of \( \sigma_n(q) \) is simply the union of the sets of roots of \( Y_n \) and \( Y_{-n} \). Furthermore, we can show that these roots are dense around the unit circle using the main theorem of Beraha et al. [6].

**Proposition 36** (Beraha et al. [6]). Suppose that

\[
P_n(q) = \sum_{j=1}^{k} \mu_j(q)\lambda_j(q)^n,
\]

(4.9)
for non-zero polynomials $\mu_1, \ldots, \mu_k, \lambda_1, \ldots, \lambda_k$. If there does not exist a constant $\omega$ such that $|\omega| = 1$ and $\lambda_j = \omega \lambda_k$ for $j \neq k$, and for some $l \geq 2$

$$|\lambda_1(x)| = |\lambda_2(x)| = \cdots = |\lambda_l(x)| > |\lambda_j(x)|,$$

for all $l + 1 \leq j \leq k$, then $x$ is a limit point of the zeroes of $\{P_n(q)\}$ — i.e., there exists a sequence $\{q_n\} \subset \mathbb{C}$ converging to $x$ such that $P_n(q_n) = 0$ for all $n$.

As each of $\alpha_n(q), \beta_n(q)$, and $\gamma_n(q)$ can be decomposed into the form described by equation (4.9) where the $\lambda_j(q)$ are simply powers of $q$, and thus have the same modulus when $q$ is on the unit circle, this immediately gives the following result.

**Corollary 37.** The roots of the families of polynomials $\{\alpha_n(q)\}, \{\beta_n(q)\}$, and $\{\gamma_n(q)\}$ are dense around the unit circle.

![Figure 4.3: Plots of the singularities of $Y_{20}(1,q)$ for the three symmetric models.](image)

As our results on convergence in Theorem 35 are only valid at points off of the unit circle, we look for singularities off the unit circle. We use a result of Konvalina and Matache [46], making use of the palindromic nature of the polynomials under consideration.

**Lemma 38** (Konvalina and Matache [45], Lemma 1). *Suppose $F(x)$ is a palindromic polynomial (its coefficient sequence is the same when read from the left or right) of degree $2N$. Then the argument of any root of $F(x)$ which lies on the unit circle satisfies*

$$\phi(\theta) = \epsilon_N + 2 \sum_{k=0}^{N-1} \epsilon_k \cos((N - k)\theta)$$
where $\epsilon_j$ denotes the coefficient of $x^j$ in $F(x)$.

Applied to our polynomials, it gives the following.

**Proposition 39.** For all natural numbers $n$, $\alpha_n(q)$ and $\gamma_n(q)$ have no roots on the unit circle, except possibly $q = \pm 1$. Furthermore, if $q$ is a root of $\beta_n(q)$ on the unit circle not equal to 1 then

$$\arg q \in \left[ \pi - \arccos \left( \sqrt{2} - \frac{1}{2} \right), \pi \right] \cup \left[ -\pi, -\pi + \arccos \left( \sqrt{2} - \frac{1}{2} \right) \right].$$

**Proof.** As $\alpha_n(q), \beta_n(q),$ and $\gamma_n(q)$ are palindromic, Lemma 38 implies, after some trigonometric simplification, that the argument of any root $q$ on the unit circle satisfies

- $\phi_A(\theta) = X + 2\cos^2(\theta) - 3$
- $\phi_B(\theta) = X^2 + (2\cos^2(\theta) - \cos(\theta) - 3)X + 2\cos^3(\theta) - 4\cos^2(\theta) - \cos(\theta) + 4$
- $\phi_C(\theta) = 2(2\cos(\theta) - 1)X^2 + (4\cos^2(\theta) - 2\cos(\theta) - 6)X + 4\cos^3(\theta) - 8\cos^2(\theta) - 6\cos(\theta) + 12$

respectively, where $X = \cos(n\theta)$. It is easy to see that $\phi_A(\theta) = 0$ only if $\theta = 0$ or $\theta = \pi$. For the other models, in order to give a bound on where the roots of each expression lie we treat $X$ as an independent real variable lying in the range $[-1, 1]$. As $\phi_B(\theta)$ and $\phi_C(\theta)$ are quadratics in $X$, it is a routine calculation to verify that $\phi_C(\theta)$ is always greater than zero unless $\theta \in \{0, \pi\}$ and that $\phi_B(\theta)$ is always greater than zero unless $\theta = 0$ or

$$\theta \in \left[ \pi - \arccos \left( \sqrt{2} - \frac{1}{2} \right), \pi \right] \cup \left[ -\pi, -\pi + \arccos \left( \sqrt{2} - \frac{1}{2} \right) \right].$$

\hfill $\square$

In fact, as the polynomials are palindromic, an application of Rouché’s theorem shows that all roots of $\alpha_n, \beta_n,$ and $\gamma_n$ converge to the unit circle as $n$ approaches infinity.

**(b) Prove the existence of the singularities**

Theoretically, it is possible that all the roots of $\sigma_n(q)$ mentioned above are extraneous, in that they are not singularities of $Y_n(q)$ but actually singularities of $Y_{-n}(q)$. Thus, we prove Lemma 40 which describes a region where we are certain to find roots of $Y_n$. Experimentally, it seems that the singularities are evenly partitioned so that those outside the unit circle belong to $Y_n$ and those inside the unit circle belong to $Y_{-n}$, but we are unable to prove this.
Lemma 40. Fix a symmetric model $A, B$ or $C$. Then $Y_n(q)$ admits at least one singularity in the complex $q$-plane minus the unit circle for an infinite number of indices $n$.

Proof. For $\arg(q) \in (-\pi/2, -3\pi/8) \cup (3\pi/8, \pi/2)$ one can verify by direct calculation that $Y_1(q) = Y_{-1}(1/q)$ when the principal branch of the square root is used in the definition of $Y_{\pm 1}(q)$. For model $A$ we see

$$Y\text{ }_{-n}(q) = \frac{(q^2 - q^{2n}) + q(q^{2n} - 1)Y_{-1}}{q^n(q^2 - 1)} = \frac{q^{1-n} - q^{n-1}}{q - q^{-1}} + \frac{q^n - q^{-n}}{q - q^{-1}Y_{-1}},$$

so that

$$Y\text{ }_{-n}(1/q) = \frac{q^{1-n} - q^{n-1}}{q - q^{-1}} + \frac{q^n - q^{-n}}{q - q^{-1}Y_1} = Y_n(q),$$

and an analogous calculation shows that $Y\text{ }_{-n}(1/q) = Y_n(q)$ for models $B$ and $C$ also. As each $Y_n(q)$ has a finite number of roots, the density of the roots on the unit circle shown in Corollary 37 then implies that $Y_n(q)$ admits a root in this region of the complex plane for an infinite number of $n$. Furthermore, Proposition 39 implies that all the singularities of $Y_n(q)$ lie off of the unit circle in this region. \hfill \square

(c) Show the singularities are distinct

We prove that the poles are distinct when they lie off of the unit circle by determining expressions for the powers of $q$ at the poles of the $Y_n(q)$.

Proposition 41. For models $A$ and $C$, if $q_n$ is a pole of $Y_n(q)$ which lies off of the unit circle then it is not a pole of $Y_k(q)$ for $k \neq n$. For model $B$, if $q_n$ is a pole of $Y_n(q)$ off of the unit circle then it is not a pole of $Y_k(q)$ for $|k - n| > 1$.

Proof. For each of the three models we find the roots of the numerators of our explicit expressions in Table 4.2 as quadratics in $q^n$. This determines functions $r_1(q)$ and $r_2(q)$, independent of $n$, such that $q_n^k = r_1(q_n)$ or $q_n^k = r_2(q_n)$ at any pole $q_n$ of $Y_n$.

Now, suppose $q_n$ is also a pole of $Y_k$ for $k \neq n$, so that $q_n^k = r_1(q_n)$ or $q_n^k = r_2(q_n)$. If $q_n^k = r_1(q_n) = q_n^n$ or $q_n^k = r_2(q_n) = q_n^n$ then it is immediate that $q_n$ must be on the unit circle. Thus we may assume, without loss of generality, that $q_n^n = r_1(q_n)$ and $q_n^k = r_2(q_n)$. We consider each model separately.

- (Model $A$) Here, $r_1(q), r_2(q) = \pm \frac{2\sqrt{-q^2}}{q^2 + \sqrt{1 - 6q^2 + q^4 - 1}}$, \hfill \hfill
so that 

\[ q_n^{n-k} = r_1(q_n)/r_2(q_n) = -1, \]

implying that \( q_n \) must lie on the unit circle.

- **(Model B)** Here,

\[ r_1(q) = \frac{2q^2}{2q^2 + \sqrt{q^4 - 2q^3 - 5q^2 - 2q + 1 - 2q + q\sqrt{q^4 - 2q^3 - 5q^2 - 2q + 1 - 1 - q^3}}, \]

and \( r_2(q) = r_1(q)/q \), so that \( q_n^{n-k} = r_1(q_n)/r_2(q_n) = q_n \). Thus, either \( n = k + 1 \) or \( q_n \) lies on the unit circle.

- **(Model C)** In this slightly trickier case we have

\[
\begin{align*}
  r_1(q) &= \frac{q(-1 - q - i\sqrt{3} + \sqrt{3}q)}{-2q^2 - \sqrt{1 - 2q - 9q^2 - 2q^3 + q^4 - 2q + q\sqrt{1 - 2q - 9q^2 - 2q^3 + q^4 + 1 + q^3}},} \\
  r_2(q) &= \frac{q(-1 - q + i\sqrt{3} - \sqrt{3}q)}{-2q^2 - \sqrt{1 - 2q - 9q^2 - 2q^3 + q^4 - 2q + q\sqrt{1 - 2q - 9q^2 - 2q^3 + q^4 + 1 + q^3]].}
\end{align*}
\]

which implies

\[ q_n^{n-k} = r_1(q_n)/r_2(q_n) = e^{-2\pi i/3} + \sqrt{3} \frac{e^{-\pi i/6}}{q - e^{\pi i/3}}. \]

Substituting \( q_n = re^{i\theta} \) into this expression allows one to see that the right hand side has modulus greater than or equal to one when \( \theta \in [0, \pi) \) and modulus less than or equal to one when \( \theta \in (-\pi, 0] \).

Suppose now that \( n > k \). If \( |q_n| < 1 \) then \( |q_n|^{n-k} < 1 \) and \( q_n \) cannot lie above the real axis (as the modulus of the right hand side would be greater than or equal to 1). Similarly, if \( |q_n| > 1 \) then \( |q_n|^{n-k} > 1 \) and \( q_n \) cannot lie beneath the real axis. As we take the principal branch of the square root in the definition of \( \Psi_n(q) \), we have \( \Psi_n(q)^* = \Psi_n(q^*) \) (where \( q^* \) denotes the complex conjugate of \( q \)) so there are, in fact, no solutions off of the unit circle or real axis. One can easily verify that there are no non-unit real solutions, and when \( n < k \) the argument is analogous.

\( \square \)

### 4.1.4 Prove that \( C(t) \) is non D-finite.

Putting these together gives the desired result.

**Theorem 42.** The generating function \( C(1, 1, t) \) is non D-finite for each of the models \( A, B, \) and \( C \).
**Proof.** By Proposition 41, for an infinite number of \( n \) there is a pole \( q_n \) of \( Y_n(q) \) which is not a pole of \( Y_k(q) \) when \( |n - k| > 1 \). We break the sum of Equation 4.8 into three parts and examine the behaviour at \( q_n \).

First consider \( A \) and \( C \). The sum is decomposed as follows:

\[
\frac{q}{1 + q^2} \cdot C(0, 1) = \sum_{k=0}^{n-2} (-1)^k Y_k Y_{k+1} + \underbrace{(-1)^n Y_n (Y_{n-1} - Y_{n+1})}_{\text{pole contribution}} + \sum_{k \geq n+1} (-1)^k Y_k Y_{k+1}. 
\]

The initial and terminal sums do not admit poles at \( q_n \) since Proposition 41 implies that \( Y_k(q) \) does not have a pole at \( q_n \) for \( k \neq n \), and the proof of Proposition 35 shows that the second summation is convergent at this point. Furthermore, if we substitute \( q_n \) into the corresponding recurrence from Table 4.2, we see \( Y_{n+1}(q_n) = -Y_n(q_n) + \epsilon \), where \( \epsilon = 0 \) for model \( A \) and \( \epsilon = 1 \) for models \( B \) and \( C \). Thus, \( Y_{n-1} - Y_{n+1} \neq 0 \) and we can conclude that \( q_n \) is a pole of the series.

For models \( A \) and \( C \), this gives an infinite number of poles of \( C(1, 1, \frac{q}{1+q^2}) \) so that it, and thus \( C(1, 1, t) \), is not D-finite. The remaining case of model \( B \) is identical, except for the fact that \( Y_n \) and \( Y_{n-1} \) share some, but not all, of their poles. If \( q_n \) is not a pole of \( Y_{n+1} \), then the argument above shows that it is a pole of \( C(1, 1, \frac{q}{1+q^2}) \). If \( q_n \) is a pole of \( Y_{n+1} \), then the summand \( (-1)^n Y_n Y_{n+1} \) has a pole of a larger order than any present in the other two summands where that pole appears, and thus cannot be canceled by the rest of the summation. This means that \( q_n \) is again a pole of \( C(1, 1, \frac{q}{1+q^2}) \), and the remainder of the argument is as for the other two cases.

**4.2 The Non-Symmetric Models**

The non-symmetric models \( D \) and \( E \) proceed in a similar manner, except that because \( C(x, y, t) \neq C(y, x, t) \) the analysis requires more complicated expressions. Once again solving
the kernel equation (after the re-parametrization \( t = q/(1 + q^2) \)) gives

**Model D:**

\[
X_\pm(y, q) = \frac{y}{2q} \cdot \left( 1 - qy + q^2 + \sqrt{q^4 - 2q^3y + (y^2 - 2)q^2 - 2qy + 1} \right)
\]

\[
Y_\pm(x, q) = \frac{x}{2q(1 + x)} \cdot \left( 1 + q^2 + \sqrt{q^4 - 4q^3x - 2q^2 + 1} \right)
\]

**Model E:**

\[
X_\pm(y, q) = \frac{y}{2q(1 + y^2)} \cdot \left( 1 - qy + q^2 + \sqrt{q^4 - 2q^3y - (3y^2 + 2)q^2 - 2qy + 1} \right)
\]

\[
Y_\pm(x, q) = \frac{x}{2q(1 + x + x^2)} \cdot \left( 1 + q^2 + \sqrt{q^4 - 2(2x^2 + 2x + 1)q^2 + 1} \right)
\]

From \( X_+(y) \) and \( Y_+(x) \) we derive two related sequences:

\[
\chi_n(x) = X_+(Y_+(\chi_{n-1}(x))), \quad \chi_0(x) = x,
\]

and

\[
\Upsilon_n(y) = Y_+(X_+(\Upsilon_{n-1}(y))), \quad \Upsilon_0(y) = y,
\]

with simple substitutions yielding the kernel relations

\[
K(\chi_n(x), Y_+(\chi_n(x))) = K(X_+(\Upsilon(y)), \Upsilon(y)) = 0.
\]

Following the methodology developed for models symmetric over the line \( y = x \), we form a telescoping sum and after some manipulation this results in an expression for the generating functions of the walks returning to each axis. For both models D and E,

\[
C \left( x, 0, \frac{q}{1 + q^2} \right) = \frac{q}{1 + q^2} \sum_{n \geq 0} \chi_n(x) \cdot \frac{(Y_+ \circ \chi_n(x) - Y_+ \circ \chi_{n-1}(x))}{\Delta_{L,n}(x)}
\]

\[
(4.10)
\]

\[
C \left( 0, y, \frac{q}{1 + q^2} \right) = \frac{q}{1 + q^2} \sum_{n \geq 0} X_+ \circ \Upsilon_n(y) \cdot \frac{(\Upsilon_n(y) - \Upsilon_{n+1}(y))}{\Delta_{R,n}(y)}
\]

\[
(4.11)
\]

The two models have identical structure in their generating function, and differ only in their respective functions \( X_+ \) and \( Y_+ \). Our greatest challenge at this point is keeping track of the various parts:

\[
\Delta_{L,n}(x) = Y_+ \circ \chi_n(x) - Y_+ \circ \chi_{n-1}(x) \quad \Delta_{R,n}(y) = \Upsilon_n(y) - \Upsilon_{n+1}(y)
\]

\[
\Delta_{L,0}(x) = Y_+(x) \quad \Delta_{R,0}(y) = \Upsilon_0(y).
\]
For each model we isolate the left and right hand sides, defining
\[ L(x, q) = qx^2C \left( x, 0, \frac{q}{1 + q^2} \right) \quad \text{and} \quad R(y, q) = qy^2C \left( 0, y, \frac{q}{1 + q^2} \right), \quad (4.12) \]
so that
\[ C \left( x, y, \frac{q}{1 + q^2} \right) = \frac{xy(1 + q^2) - L(x, q) - R(y, q)}{K(x, y)}, \]
and the counting generating function has the form
\[ C \left( 1, 1, \frac{q}{1 + q^2} \right) = \frac{(1 + q^2) - L(1, q) - R(1, q)}{K(1, 1)}. \quad (4.13) \]
For both asymmetric models we find an infinite set of points on the imaginary axis at which \( L(1, q) \) is singular, but \( R(1, q) \) is convergent. Similar to the previous cases, we use the coefficients of \( K(x, y) \) to form recurrences for the functions under consideration:
\[
\bar{\chi}_n = (q + 1/q) \bar{Y}_+ \circ \bar{\chi}_{n-1} - \bar{\chi}_{n-1} - 1, \quad \bar{Y}_n = (q + 1/q) \bar{X}_+ \circ \bar{Y}_{n-1} - \bar{Y}_{n-1}, \quad \bar{X}_+ \circ \bar{Y}_n = (q + 1/q) \bar{X}_n - \bar{X}_+ \circ \bar{Y}_{n-1} - 1. \quad (4.14)
\]
Solving these recurrences yields the closed form expressions found in Table 4.4.

<table>
<thead>
<tr>
<th>Function</th>
<th>Explicit Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{\chi}_n )</td>
<td>( \frac{(q^{4n+3} - q^{4n+1} - q^3 + q)\bar{Y}_+ - 2q^{4n+2} + q^4 + 2q^{2n+2} + q^2 - 2q^2}{q^{2n}(q^2 - 1)^2} )</td>
</tr>
<tr>
<td>( \bar{Y}_+ \circ \bar{\chi}_n )</td>
<td>( \frac{(q^{4n+4} - q^{4n+2} - q^2 + 1)\bar{Y}_+ - 2q^{4n+3} + q^{4n+1} + q^{2n+3} + q^{2n+1} + q^3 - 2q}{q^{2n}(q^2 - 1)^2} )</td>
</tr>
<tr>
<td>( \bar{Y}_n )</td>
<td>( \frac{(q^{4n+3} - q^{4n+1} - q^3 + q)\bar{X}_+ - q^{4n+2} - q^{4n+1} + q^{4n} + q^{2n+3} + q^{2n+1} + q^4 - q^3 - q^2}{q^{2n}(q^2 - 1)^2} )</td>
</tr>
<tr>
<td>( \bar{X}_+ \circ \bar{Y}_n )</td>
<td>( \frac{(q^{4n+4} - q^{4n+2} - q^2 + 1)\bar{X}_+ - q^{4n+3} - q^{4n+2} + q^{4n+1} + 2q^{2n+2} + q^3 - q^2 - q}{q^{2n}(q^2 - 1)^2} )</td>
</tr>
</tbody>
</table>

Table 4.4: Solutions to the recurrences in Equation 4.14.

The analysis proceeds in the same vein as the arguments for the previous models, made harder by the fact that there are two infinite series present in our representation of the generating function but at the same time made simpler by the fact that there are now an infinite number of singularities of the generating function lying on the imaginary axis.

To begin, one shows that for each \( n \) the function \( \chi_{2n}(ri) \) has a distinct singularity for some \( r \in (1, 2) \). The function \( L(1, q) \) is singular at each of these points, while \( R(1, q) \) is
analytic, and it can thus be proven that each point gives a singularity of $C \left(1, 1, \frac{q}{1+q^2}\right)$. The arguments use Gröbner Basis calculations to show the existence of or bound the real zeroes of four polynomials whose roots include the singularities of $\chi_n(ri)$, $(Y_+ \circ \chi_n)(ri)$, $Y_n(ri)$, and $(X_+ \circ Y_n)(ri)$, respectively.

**Theorem 43** (Melczer and Mishna [50], Theorems 18 and 22). The generating function $C(1,1,t)$ is non D-finite for the models $\mathcal{D}$ and $\mathcal{E}$.

The representation of the generating function given by Equation 4.13 also allows for a calculation of dominant asymptotics.

**Theorem 44** (Mishna and Rechnitzer [52], Proposition 16). For the model $\mathcal{D}$,

$$c_n = \kappa_D \frac{3^n}{\sqrt{n}} (1 + o(1)), \quad \text{where} \quad \kappa_D \in \left[0, \sqrt{\frac{3}{\pi}}\right].$$

**Theorem 45** (Melczer and Mishna [50], Corollary 15). For the model $\mathcal{E}$,

$$c_n = \kappa_E \cdot 4^n + O \left((1 + 2\sqrt{2})^n\right),$$

where $\kappa_E \in \left[\frac{122}{525}, \frac{7}{10}\right]$.

### 4.3 Other Methods for Non D-Finiteness

Unfortunately, the iterated kernel method relies heavily on the fact that one of the roots of the kernel as a quadratic in $x$ (respectively in $y$) has an expansion $X_+(y) = yt + yO(t^2)$ (respectively $Y_+(x) = xt + xO(t^2)$) which makes the infinite series representations of $C(1,1,t)$ in Equations (4.6) and (4.13) valid. This restricts the method to work only (amongst those with unit steps) on the five singular models, and it is still unknown whether the generating functions $C(1,1,t)$ counting the total number of walks in the quarter plane for the other 51 models with infinite group are non D-Finite.
In addition to the computational evidence discussed in Section 3.3.3, we now mention two other results which suggest (but do not prove) that the generating functions $C(t) = C(1, 1, t)$ are non D-finite for the 51 non-singular walks with infinite group. Bostan et al. [14] proved that the generating function $C(0, 0, t)$, which counts the number of walks in each model which return to the origin, is not D-finite (the singular models have no walks returning to the origin and thus $C(0, 0, t)$ is trivially D-finite). Recall that a result of Garoufalidis [36] on the nature of G-series shows that if the generating function $C(0, 0, t)$ is D-finite then the coefficient sequence $a_n = [t^n]C(0, 0, t)$ must grow asymptotically as $a_n \sim \kappa \rho^n n^\alpha (\log n)^\beta$, where $\alpha \in \mathbb{Q}$. The authors use recent work of Denisov and Wachtel [30] to represent the singular exponent of each non-singular model as $\alpha = 1 - \pi/\arccos(c)$, where $c$ is an algebraic number whose minimal polynomial $m_c(x, y)$ can be determined via elimination theory from the characteristic polynomial $P(x, y)$ and its partial derivatives. If $\arccos(c)/\pi$ were a rational number, then $c$ must have the form $c = (z + 1/z)/2$ for $z$ a root of unity. The authors conclude their proof of non D-finiteness by showing algorithmically that for each model the explicit Laurent polynomial $m_c((z + 1/z)/2)$ is nonzero at any root of unity.

**Theorem 46** (Bostan et al. [14], Theorem 1). For any of the 51 non-singular models, the generating function $C(0, 0, t)$, and thus also the tri-variate generating function $C(x, y, t)$, is not D-finite.

In addition, Kurkova and Raschel [48] show that that the tri-variate generating functions $C(x, y, t)$ marking the endpoint of a walk are not D-finite for each non-singular walk by solving boundary value problems of Riemann-Carleman type to obtain integral expressions for the generating functions $C(x, 0, t)$, $C(0, y, t)$, and $C(0, 0, t)$. For certain fixed $t_0 \in (0, 1/|S|)$, these integral representations give a construction of $C(x, 0, t_0)$ and $C(0, y, t_0)$ as multi-valued meromorphic functions and prove that they admit an infinite number of singularities (the sets of singularities are even dense on certain curves). Working in a similar fashion, Fayolle and Raschel [31] used integral representations of the generating functions for walks in the quarter plane to give a general approach for calculating the asymptotics of all non-singular models. They linked the asymptotic growth of a model to combinatorial properties of its step set such as its vector sum (called its drift), and although their methods do not apply to the singular models the asymptotics worked out for the singular models in Melczer and Mishna [50] fits into what is predicted by their template.
Theorem 47 (Kurkova and Raschel [48], Theorem 1). For any of the 51 non-singular models, the generating functions $C(x,0,t_0)$ and $C(0,y,t_0)$ are not D-finite for an infinite number of $t_0 \in (0,1/|S|)$. Therefore, the tri-variate generating function $C(x,y,t)$ is also not D-finite.
Chapter 5

Highly Symmetric Walks in an Orthant

As noted in Chapter 3, studying a lattice path model of dimension two or higher requires algebraic and geometric properties of the kernel equation related to the model. A lack of general methods for dealing with these complications has led to the majority of the results in this area focusing on the simplest case of two dimensional walks restricted to a quadrant. Recently, advances in the field of analytic combinatorics in several variables (detailed in Pemantle and Wilson [55]) permit a more systematic analysis. We take advantage of them here, using the techniques in a general way to create a new framework giving the first non-trivial asymptotic results for models in arbitrary dimensions.

In this chapter we demonstrate an appropriate group for the higher dimensional models whose step sets are symmetric with respect to each axis. We use this group in the orbit sum method to write the generating function for the number of walks restricted to an orthant as the diagonal of a multivariate rational function. This proves the generating function is D-finite (which is not surprising given previous results relating the D-finiteness of lattice path models to symmetries in their step sets, for instance the work of Bousquet-Mélou [15]). The symmetries present in the step sets of these models simplify the geometry of the surfaces which dictate asymptotic behaviour of the corresponding generating functions – allowing access to dominant asymptotics for the number of such walks. Verifying the conditions needed to apply the results of Pemantle and Wilson [55] is non-trivial, especially in general settings, however in the case of highly symmetric models we can and do perform the analysis.
To be precise, the lattice path models we consider have, for a fixed dimension $d$, restricting region $R = \mathbb{N}^d \subseteq \mathbb{Z}^d$ and the following constraints on their step sets. We say that the step set $\mathcal{S} \subseteq \{\pm 1, 0\}^d \setminus \{0\}$ is symmetric about the $z_k$ axis if $(i_1, \ldots, i_k, \ldots, i_d) \in \mathcal{S}$ implies that $(i_1, \ldots, -i_k, \ldots, i_d) \in \mathcal{S}$. We deal only with step sets which are symmetric about every axis, and assume that a step set is non-degenerate in the sense that for each coordinate $\mathcal{S}$ contains at least one step which moves forward in that coordinate (if not then it is isomorphic to a non-degenerate symmetric model in a lower dimension). These are the highly symmetric models. There are four such non-isomorphic models when $d = 2$ – listed in Table 5.1 – and 109 (possibly isomorphic) models when $d = 3$.

After writing the generating function $C(t)$ as the (complete) diagonal of a multivariate rational function, we apply methods for the asymptotic enumeration of rational diagonals developed by Pemantle and Wilson [54] and Raichev and Wilson [56]. Following their procedures, we study the surface comprised of the points where the denominator of this rational function vanishes – called the singular variety of the rational function – to determine the desired asymptotics. The condition of having a symmetry across each axis ensures that the variety is smooth and allows us to calculate the leading asymptotic term explicitly.
CHAPTER 5. HIGHLY SYMMETRIC WALKS IN AN ORTHANT

Asymptotics

\[ \frac{4}{\pi \sqrt{1 \cdot 1}} \cdot n^{-1} \cdot 4^n = \frac{4}{\pi} \cdot \frac{4^n}{n} \]

\[ \frac{6}{\pi \sqrt{3 \cdot 2}} \cdot n^{-1} \cdot 6^n = \frac{\sqrt{6}}{\pi} \cdot \frac{6^n}{n} \]

\[ \frac{4}{\pi \sqrt{2 \cdot 2}} \cdot n^{-1} \cdot 4^n = \frac{2}{\pi} \cdot \frac{4^n}{n} \]

\[ \frac{8}{\pi \sqrt{3 \cdot 3}} \cdot n^{-1} \cdot 8^n = \frac{8}{3\pi} \cdot \frac{8^n}{n} \]

Table 5.1: The four highly symmetric models with unit steps in the quarter plane.

Although the methods also apply in more general cases where the singular variety is non-smooth, the results are less automatic and thus harder to manage in such a general setting. The main result of this chapter is the following.

**Main Theorem.** Let \( S \subseteq \{-1, 0, 1\}^d \setminus \{0\} \) be a set of unit steps in dimension \( d \). If \( S \) is symmetric with respect to each axis, and \( S \) takes a positive step in each direction, then the number of walks of length \( n \) taking steps in \( S \), beginning at the origin, and never leaving the positive orthant has asymptotic expansion

\[ c_n \sim \left[ \sum_{k} s^{(k)} \right]^{-1/2} \pi^{-d/2} |S|^{d/2} \cdot n^{-d/2} \cdot |S|^n, \]

where \( s^{(k)} \) denotes the number of steps in \( S \) which have \( k \)th coordinate 1.

This formula is easy to compute explicitly for any given model, and for certain infinite families as well.

**Example 48.** Let \( S = \{-1, 0, 1\}^d \setminus \{0\} \), the full set of possible steps, which is symmetric across each axis. We compute that \( |S| = 3^d - 1 \), and \( s^{(j)} = 3^{d-1} \) for all \( j \) and so

\[ c_n \sim \left( \frac{3^d - 1}{3^d (d-1)^{d/2}} \right) \cdot n^{-d/2} \cdot (3^d - 1)^n. \]

**Example 49.** Let \( e_k = (0, \ldots, 0, 1, 0, \ldots, 0) \) be the \( k \)th standard basis vector in \( \mathbb{R}^d \), and define \( S = \{e_1, -e_1, \ldots, e_d, -e_d\} \). Then the number of walks of length \( n \) taking steps from \( S \) and never leaving the positive orthant has asymptotic expansion

\[ c_n \sim \left( \frac{2d}{\pi} \right)^{d/2} \cdot n^{-d/2} \cdot (2d)^n. \]
Example 50. When \( d = 2 \) there are four highly symmetric walks in the quarter plane, listed in Table 5.1. Applying Theorem 60 verifies the asymptotic results guessed by Bostan and Kauers [10] and listed in Table 3.1 of Chapter 3. See Table 5.1 for the required calculations.

We start by applying an extension of the orbit sum method to find expressions for the generating functions of walks in the positive orthant. This is converted into a representation of \( C(t) \) as the diagonal of a rational power series in multiple variables, on which the methods of Pemantle and Wilson [55] apply to obtain Theorem 60.

5.1 The Orbit Sum Method in Higher Dimensions

Suppose \( S \) is a highly symmetric model. In this section we derive a functional equation for a multivariate generating function, apply the orbit sum method to derive a closed expression related to this generating function, and conclude by writing the univariate counting generating function for the number of walks as the complete diagonal of a rational function.

5.1.1 A Functional Equation

To begin, we (as usual) define the multivariate generating function:

\[
C(z, t) = \sum_{n \geq 0} \left( \sum_{i \in \mathbb{Z}^d} c_{i,n} z_1^{i_1} \cdots z_d^{i_d} \right) t^n \in \mathbb{Q}[z_1, z_1, \ldots, z_d, \bar{z}_d][[t]],
\]

where \( c_{i,n} \) counts the number of walks of length \( n \) which end at the point \( i \in \mathbb{Z}^d \), taking steps from \( S \) and staying in the positive orthant. Again, the series \( C(t) := C(1, t) \) is the generating function for the total number of walks in the orthant, and we can recover the series for walks ending on the hyperplane \( z_k = 0 \) by setting \( z_k = 0 \) in the series \( C(z, t) \). The characteristic polynomial is

\[
P(z) = \sum_{(s_1, s_2, \ldots, s_d) \in S} z_1^{s_1} \cdots z_d^{s_d} \in \mathbb{Q}[z_1, \bar{z}_1, \ldots, z_d, \bar{z}_d].
\]

As in the cases covered in Chapter 3, we have the recursive definition that a walk of length \( n + 1 \) is a walk of length \( n \) followed by a step in \( S \). To ensure the condition that the walks remain in the positive orthant, we must not count walks that add a step with
a negative $k$-th component to a walk ending on the hyperplane $z_k = 0$. Just as in the
kernel equation (3.6) for models in the quarter plane, it is sufficient to subtract a term
of the form $t \bar{z}_k C(z_1, \ldots, z_{k-1}, 0, z_{k+1}, \ldots, z_d, t)$, and if a given step has several negative
components we must use inclusion and exclusion to prevent over compensation.

To be explicit, we note that by our assumptions on $S$ for each coordinate there exists
some step in $S$ with $-1$ in that coordinate. Translating the combinatorial recurrence de-
scribed above via the Principle of Inclusion and Exclusion implies that $C(z, t)$ satisfies the
functional equation

$$(z_1 \cdots z_d) C(z, t) = (z_1 \cdots z_d) + (z_1 \cdots z_d) P(z) C(z, t)$$

$$- \sum_{V \subseteq [d]} (-1)^{|V|} (z_1 \cdots z_d) P(z) C(z, t) \bigg|_{\{z_j = 0 : j \in V\}},$$

(5.1)

where $[d] = \{1, \ldots, d\}$ and we note that $(z_1 \cdots z_d) P(z) C(z, t)$ is evaluated at $z_k = 0$ by
simplifying the expression before substitution (as the lowest power of $z_k$ to appear in the
Laurent polynomial $P(z)$ is -1). This implies that the kernel equation here has the form

$$(z_1 \cdots z_d) (1 - tP(z)) C(z, t) = (z_1 \cdots z_d) + \sum_{k=1}^d A_k(z_1, \ldots, z_{k-1}, z_{k+1} \ldots, z_d, t),$$

(5.2)

where each $A_k \in \mathbb{Q}[z_1, \ldots, z_{k-1}, z_{k+1} \ldots, z_d][[t]].$

**Example 51.** Set $S = \{e_1, -e_1, \ldots, e_d, -e_d\}$. In this case $P(z) = \sum_{j=1}^d (z_j + \bar{z}_j)$, so
$(z_1 \cdots z_d) P(z)$ vanishes when at least two of the $z_j$ are zero, and the generating function
satisfies

$$C(z, t) = 1 + tP(z) C(z, t) - \sum_{j=1}^d \bar{z}_j t C(z_1, \ldots, z_{j-1}, 0, z_{j+1} \ldots, z_d).$$

\(<\)

### 5.1.2 The Orbit Sum

For any $d$-dimensional model, we define the group $\mathcal{G}$ of $2^d$ rational maps by

$$\mathcal{G} := \left\{ (z_1, \ldots, z_d) \mapsto (z_1^{i_1}, \ldots, z_d^{i_d}) : (i_1, \ldots, i_d) \in \{-1, 1\}^d \right\}.$$

(5.3)

Given $\sigma \in \mathcal{G}$, we can consider $\sigma$ as a map on $\mathbb{Q}[z_1, \bar{z}_1, \ldots, z_d, \bar{z}_d][[t]]$ through the group
action $\sigma(F(z, t)) := F(\sigma(z), t)$; due to the symmetry of the step set across each axis, one
can verify that $\sigma(P(z)) = P(\sigma(z)) = P(z)$ always holds. The fact that this group does not
depend on the step set of the model – only on the dimension \( d \) – is crucial to obtaining the general results in this chapter. When \( d \) equals two, the group \( \mathcal{G} \) matches with the group as defined by Fayolle et al. [32] and Bousquet-Mélou and Mishna [18].

We now apply each element of \( \mathcal{G} \) to Equation (5.2), and take a telescoping sum. Define \( \text{sgn}(\sigma) = (-1)^r \), where \( r = \# \{ k : \sigma(z_k) = z_k \} \), and let \( \sigma_k \) be the map which sends \( z_k \) to \( z_k \) and fixes all other components of \((z_1, \ldots, z_d)\).

**Lemma 52.** Let \( C(z, t) \) be the generating function counting the number of walks of length \( n \) with marked endpoint. Then, as elements of the ring \( \mathbb{Q}[z_1, z_2, \ldots, z_d, \bar{z}_d][[t]] \),

\[
\sum_{\sigma \in \mathcal{G}} \text{sgn}(\sigma) \cdot \sigma(z_1 \cdots z_d)C(z, t) = \frac{\sum_{\sigma \in \mathcal{G}} \text{sgn}(\sigma) \cdot \sigma(z_1 \cdots z_d)}{1 - t P(z)}.
\] (5.4)

**Proof.** For each \( \sigma \in \mathcal{G} \) we have \( \text{sgn}(\sigma) = -\text{sgn}(\sigma_k \sigma) \) and, as \( A_k(z_1, \ldots, z_k-1, z_k+1, \ldots, z_d) \) does not contain a \( z_k \)-term,

\[
\sigma(A_k(z_1, \ldots, z_k-1, z_k+1 \ldots, z_d, t)) = (\sigma_k \sigma)(A_k(z_1, \ldots, z_k-1, z_k+1 \ldots, z_d, t)).
\]

Thus, applying each \( \sigma \in \mathcal{G} \) to Equation (5.2) and summing the results weighted by \( \text{sgn}(\sigma) \) cancels all \( A_k \) terms. Algebraic manipulations, together with the fact that the group elements fix \( P(z_1, \ldots, z_d) \), then give Equation (5.4).

Note that each term in the expansion of

\[
\sigma_1(z_1, \ldots, z_d)\sigma_1(C(z, t)) = -(\bar{z}_1 z_2 \cdots z_d)C(\bar{z}_1, z_2, \ldots, z_d, t) \in \mathbb{Q}[z_1, \bar{z}_1, \ldots, z_d, \bar{z}_d][[t]]
\]

has a negative power of \( z_1 \). In fact, except for when \( \sigma \) is the identity any term in the expansion of each \( \sigma(z_1 \cdots z_d)\sigma(F(z_1, \ldots, z_d, t)) \) term in the left hand side of Equation (5.4) contains a negative power of at least one variable. **Lemma 53** follows immediately from the identity

\[
\sum_{\sigma \in \mathcal{G}} \text{sgn}(\sigma) \cdot \sigma(z_1 \cdots z_d) = (z_1 - \bar{z}_1) \cdots (z_d - \bar{z}_d),
\]

which can be proven by induction.

**Lemma 53.** Let \( C(z, t) \) be the generating function counting the number of walks of length \( n \) with marked endpoint. Then

\[
C(z, t) = [z_1^2] \cdots [z_d^2] R(z, t),
\] (5.5)
where
\[ R(z, t) = \frac{(z_1 - \bar{z}_1) \cdots (z_d - \bar{z}_d)}{(z_1 \cdots z_d)(1 - tP(z))}. \]

### 5.1.3 The Generating Function as a Diagonal

A positive series extraction can be written as a Hadamard product with a geometric series: given \( B(z, t) \in \mathbb{Q}[z_1, z_1, \ldots, z_d, \bar{z}_d][[t]] \) it follows from a direct calculation that
\[
[z_1^\geq] \cdots [z_d^\geq] B(z, t) = \left( \frac{1}{(1 - z_1)(1 - \bar{z}_2) \cdots (1 - z_d)} \right) \odot B(z, t)
\]
Consequently, as discussed in Chapter 2 these generating functions can be written as diagonals of rational functions in \( 2d \) variables, and Theorem 16 shows that the generating functions are D-finite.

**Proposition 54.** Under our restrictions on \( S \), the generating function \( C(z, t) \) is D-finite. In particular, the univariate generating function \( C(t) = C(1, t) \) is D-finite.

In fact, the univariate generating function \( C(t) \) can be expressed as the diagonal of a rational function in \( d \) variables.

**Proposition 55.** Let \( B(z, t) \) be an element of \( \mathbb{Q}[z_1, \bar{z}_1, \ldots, z_d, \bar{z}_d][[t]] \), and suppose that \( B(y_1, \ldots, y_d, y_1 \cdots y_d \cdot t) \) is a power series in the variables \( y_1, \ldots, y_d, t \). Then
\[
[z_1^\geq] \cdots [z_d^\geq] B(z, t) \bigg|_{z_1=1,\ldots,z_d=1} = \Delta \left( \frac{B(y_1, \ldots, y_d, y_1 \cdots y_d \cdot t)}{(1 - y_1) \cdots (1 - y_d)} \right)
\]

**Proof.** Suppose that \( B \) has the expansion
\[
B(z_1, \ldots, z_d, t) = \sum_{n \geq 0} \left( \sum_{i \in \mathbb{Z}^d} b_{i,n} z_1^{i_1} \cdots z_d^{i_d} \right) t^n.
\]
Then the right hand side of the above equation is given by
\[
\Delta \left( \sum_{k \geq 0} y_1^{i_1} \cdots \sum_{k \geq 0} y_d^{i_d} \right) \left( \sum_{i,n} b_{i,n} y_1^{n-i_1} \cdots y_d^{n-i_d} t^n \right),
\]
so that the coefficient of \( t^n \) in the diagonal is the sum of all terms \( b_{i_1,\ldots, i_d,n} \) with \( i_1, \ldots, i_d \geq 0 \) (by assumption there are only finitely many which are non-zero). But this is exactly the coefficient of \( t^n \) on the left hand side. \( \Box \)
Combining Lemma 53 and Proposition 55, we see that the generating function for the total number of walks of length \( n \) has the representation

\[
C(t) = \Delta \left( \frac{G(y,t)}{H(y,t)} \right)
\]

where

\[
G(y,t) = \frac{(1 - y_1^2) \cdots (1 - y_d^2)}{1 - t(y_1 \cdots y_d)P(y)}\left(1 - \frac{1}{1 - t(y_1 \cdots y_d)P(y)}\right) = \frac{(1 + y_1) \cdots (1 + y_d)}{1 - t(y_1 \cdots y_d)P(y)}
\]

(5.6)

To be precise \( G(y,t) \) and \( H(y,t) \) are defined as the numerator and denominator of Equation (5.6).

**Example 56.** For the walks defined by \( S = \{e_1, -e_1, \ldots, e_d, -e_d\} \), we have

\[
\frac{G(y,t)}{H(y,t)} = \frac{(1 + y_1) \cdots (1 + y_d)}{1 - t \sum_{k=1}^n \frac{1}{(1 + y_k^2)(y_1 \cdots y_k - y_{k+1} \cdots y_d)}}.
\]

Note that this rational function is not unique, in the sense that there are other rational functions whose diagonals yield the same counting sequence.

5.1.4 The Singular Variety

Here, we pause to note that the combinatorial symmetries of the step sets that we consider affect the geometry of the variety (complex set of zeroes) of \( H(y,t) \) – called the singular variety. This has a direct impact on both the asymptotics of the counting sequence under consideration and the ease with which its asymptotics are computed. In particular, the factors of the form \((1 - y_k)\) present in the denominator of this rational function before simplification would have made the singular variety non-smooth. Although non-smooth varieties can be handled in many cases – see Pemantle and Wilson [55] – having a smooth singular variety is the easiest situation in which one can work in the multivariate setting. As discussed in the concluding chapter of this thesis, understanding the interplay between the step set symmetry and the singular variety geometry (and in the process dealing with the non-smooth cases) is promising future work.

5.2 Analytic Combinatorics in Several Variables

Next we determine an asymptotic estimate for the diagonal of the multivariate power series \( \frac{G(y,t)}{H(y,t)} \) by studying the variety (complex set of zeroes) \( \mathcal{V} \subseteq \mathbb{C}^{d+1} \) of the denominator

\[
H(y,t) = 1 - t(y_1 \cdots y_d)P(y).
\]
In this more complicated situation, a particular set of singular points – called the critical points – which could affect the asymptotics of $\Delta(G/H)$ are first computed. The set of critical points is then refined to those which determine the dominant asymptotics up to an exponential decay; this refined set is called the set of minimal points as they are the critical points which are ‘closest’ to the origin in a sense made precise below. The asymptotic results themselves come from calculating a Fourier-Laplace type integral, and after determining the minimal points we find the asymptotics using pre-computed formulas for such integrals which can be found in Pemantle and Wilson [55].

To begin, we verify our claim in the previous section that the variety is smooth – i.e., that at every point on $V$ one of the partial derivatives $H_{y_k}$ or $H_t$ does not vanish. Indeed, any non-smooth point on $V$ would have to satisfy both

$$1 - t(y_1 \cdots y_d)P(y) = H = 0$$
and
$$-(y_1 \cdots y_d)P(y) = H_t = 0,$$

which can never occur. Equivalently, this shows that around each point in $V$ there exists a neighborhood $N \subseteq \mathbb{C}^{d+1}$ such that $V \cap N$ is a complex submanifold of $N$.

### 5.2.1 Critical Points

The next step is to determine the critical points. Coming from stratified Morse theory, in the smooth setting they are precisely the points which satisfy the critical point equations

$$H = 0, \quad tH_t = y_1 H_{y_1}, \quad tH_t = y_2 H_{y_2}, \quad \ldots \quad tH_t = y_d H_{y_d}. \tag{5.7}$$

Given $y = (y_1, \ldots, y_d) \in \mathbb{C}^d$, we define $y_{\overline{k}} := (y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_d) \in \mathbb{C}^{d-1}$. As each step in $S$ has coordinates taking values in $\{-1, 0, 1\}$, we may collect the coefficients of the $k^{\text{th}}$ variable, and use the symmetry of the step set to write

$$P(y) = (\overline{y}_{k} + y_{k})P_1^{(k)}(y_{\overline{k}}) + P_0^{(k)}(y_{\overline{k}}), \tag{5.8}$$

which uniquely defines the Laurent polynomials $P_1^{(k)}(y_{\overline{k}})$ and $P_0^{(k)}(y_{\overline{k}})$. Looking at the equation involving $H_{y_k}$, we see that at any critical point,

$$t(y_1 \cdots y_d)P(y) = t(y_1 \cdots y_d)P(y) + t(y_1 \cdots y_d)(y_k P_{y_k}(y)),$$
which implies
\[
0 = t(y_1 \cdots y_d) \cdot y_k P_{y_k}(y)
= t \left( y_k^2 - 1 \right) (y_1 \cdots y_{k-1} y_{k+1} \cdots y_d) P_{1}^{(k)}(y_k).
\] (5.9)

Note that while \((y_1 \cdots y_{k-1} y_{k+1} \cdots y_d) P_{1}^{(k)}(y_k)\) is a polynomial, \(P_{1}^{(k)}(y_k)\) itself is a Laurent polynomial, so one must be careful when specializing variables to 0 in the expression. This calculation characterizes the critical points on \(\mathcal{V}\).

**Proposition 57.** The point \((z, t) = (z_1, \ldots, z_d, t) \in \mathcal{V}\) is a critical point of \(\mathcal{V}\) if and only if for each \(1 \leq k \leq d\) either:

- \(z_k = \pm 1\) or,
- the polynomial \((y_1 \cdots y_{k-1} y_{k+1} \cdots y_d) P_{1}^{(k)}(y_k)\) has a root at \(z_k\).

**Proof.** We have shown above that the critical point equations reduce to Equation (5.9). Furthermore, if \(t\) were zero at a point on \(\mathcal{V}\) then \(0 = H(z_1, \ldots, z_n, 0) = 1\), a contradiction. \(\square\)

### 5.2.2 Minimal Points

Among the critical points, only those which are ‘closest’ to the origin will contribute to the asymptotics, up to an exponentially decaying error. This is analogous to the univariate case, where the singularities of minimum modulus are those which contribute to the dominant asymptotic term. To be precise, for any point \((z, t) \in \mathbb{C}^{d+1}\) we define the torus
\[
T(z, t) := \{(w, t') \in \mathbb{C}^{d+1} : |t'| \leq |t| \text{ and } |w_j| \leq |z_j| \text{ for } j = 1, \ldots, d\}.
\]

The critical point \((z, t)\) is called **strictly minimal** if \(T(z, t) \cap \mathcal{V} = \{(z, t)\}\), and **finitely minimal** if the intersection contains only a finite number of points, all of which are on the boundary of \(T(z, t)\). Finally, we call a critical point \(z\) **isolated** if there exists a neighbourhood of \(\mathbb{C}^{d+1}\) where it is the only critical point. In our case, we need only be concerned with isolated finitely minimal points.

**Proposition 58.** The point \(\rho = (1, 1/|S|)\) is a finitely minimal point of the variety \(\mathcal{V}\). Furthermore, any point in \(T(\rho) \cap \mathcal{V}\) is an isolated critical point.
Proof. The point $\rho$ is critical as it lies on $V$ and its first $d$ coordinates are all one. Suppose $(w, t_w) \in V \cap T(\rho)$, where we note that any choice of $w$ uniquely determines $t_w$ on $V$. Then, as $t_w \neq 0$,
$$\left| \sum_{(i_1, \ldots, i_d) \in S} w_{i_1+1} w_{i_2+1} \cdots w_{i_d+1} \right| = \left| w_1 \cdots w_d P(w) \right| = \left| \frac{1}{t_w} \right| \geq |S|.$$  
But $(w, t_w) \in T(\rho)$ implies $|w_j| \leq 1$ for each $1 \leq j \leq d$. Thus, the above inequality states that the sum of $|S|$ complex numbers of modulus at most one has modulus $|S|$. The only way this can occur is if each term in the sum has modulus one, and all terms point in the same direction in the complex plane. By symmetry, and the assumption that we take a positive step in each direction, there are two terms of the form $w_{i_2+1} w_{i_3+1} \cdots w_{i_d+1}$ and $w_{i_2} w_{i_3+1} \cdots w_{i_d+1}$ in the sum, so that $w_1^2$ must be 1 in order for them to point in the same direction. This shows $w_1 = \pm 1$, and the same argument applies to each $w_k$, so there are at most $2^d$ points in $V \cap T(\rho)$.

By Proposition 57 every such point $(w, t_w) \in V \cap T(\rho)$ is critical, and to show it is isolated it is sufficient to prove $P^{(k)}(1_{w_\overline{\rho}}) \neq 0$ for all $1 \leq k \leq d$. Indeed, if $P^{(k)}(w_\overline{\rho}) = 0$ then $w \in V$ implies
$$0 = 1 - t_w w_1 \cdots w_d P(w) = 1 - t_w w_1 \cdots w_d P^{(k)}(w_\overline{\rho}).$$
Thus,
$$|t_w| = \frac{1}{w_1 \cdots w_d P^{(k)}(w_\overline{\rho})} \geq \frac{1}{P^{(k)}(w_\overline{\rho})} \geq \frac{1}{P^{(k)}(1)} > \frac{1}{|S|},$$
by our assumption that $S$ contains a step which moves forward in the $k$th coordinate. This contradicts $(w, t_w) \in T(\rho)$.

5.2.3 Asymptotics Results

On all of $V$ we may parametrize the coordinate $t$ as
$$t(y) = \frac{1}{y_1 \cdots y_d P(y)}.$$
CHAPTER 5. HIGHLY SYMMETRIC WALKS IN AN ORTHANT

For each point \((w, t_w) \in V \cap T(\rho)\), the analysis of Pemantle and Wilson [54] shows that the asymptotics of such an integral depends on the function

\[
\tilde{f}^{(w)}(\theta) = \log \left( \frac{l(w_1 e^{i\theta_1}, \ldots, w_d e^{i\theta_d})}{t_w} \right) + i \sum_{k=1}^{d} \theta_k
\]

\[
= \log \left( \frac{P(w)}{e^{i(\theta_1 + \cdots + \theta_d)} P(w_1 e^{i\theta_1}, \ldots, w_d e^{i\theta_d})} \right) + i(\theta_1 + \cdots + \theta_d)
\]

\[
= \log P(w) - \log (w_1 e^{i\theta_1}, \ldots, w_d e^{i\theta_d}).
\] (5.10)

Let \(H_w\) denote the determinant of the Hessian of \(\tilde{f}^{(w)}(\theta)\) at \(0\):

\[
H_w := \begin{vmatrix}
\tilde{f}_{\theta_1 \theta_1}^{(w)}(0) & \tilde{f}_{\theta_1 \theta_2}^{(w)}(0) & \cdots & \tilde{f}_{\theta_1 \theta_d}^{(w)}(0) \\
\tilde{f}_{\theta_2 \theta_1}^{(w)}(0) & \tilde{f}_{\theta_2 \theta_2}^{(w)}(0) & \cdots & \tilde{f}_{\theta_2 \theta_d}^{(w)}(0) \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{f}_{\theta_d \theta_1}^{(w)}(0) & \tilde{f}_{\theta_d \theta_2}^{(w)}(0) & \cdots & \tilde{f}_{\theta_d \theta_d}^{(w)}(0)
\end{vmatrix}
\]

If \(H_w \neq 0\), then we say \((w, t_w)\) is \textit{non-degenerate}. The main asymptotic result of smooth multivariate analytic combinatorics, in this restricted context, is the following (the original result allows for asymptotic expansions of coefficient sequences more generally defined from multivariate meromorphic functions than the diagonal sequence).

**Theorem 59** (Adapted from Theorem 3.5 of Pemantle and Wilson [54]). \textit{Suppose that} \(F(y, t) = G(y, t)/H(y, t)\) \textit{has an isolated strictly minimal simple pole at} \((z, t_z)\). \textit{If} \(tH_t\) \textit{does not vanish at} \((z, t_z)\) \textit{then there is an asymptotic expansion}

\[
c_n \sim (z_1 \cdots z_d \cdot t)^{-n} \sum_{l \geq l_0} C_l n^{-(d+l)/2}
\] (5.11)

\textit{for constants} \(C_l\), \textit{where} \(l_0\) \textit{is the degree to which} \(G\) \textit{vanishes near} \((z, t_z)\). \textit{When} \(G\) \textit{does not vanish at} \((z, t_z)\) \textit{then} \(l_0 = 0\) \textit{and the leading term of this expansion is}

\[
C_0 = (2\pi)^{-d/2} H_z^{-1/2} \cdot \frac{G(z, t_z)}{tH_t(z, t_z)}.
\] (5.12)

In fact, Corollary 3.7 of Pemantle and Wilson [54] shows that in the case of a finitely minimal point one can simply sum the contributions of each point. Combining this with our above calculations gives the main result of this chapter.
Theorem 60. Let $S \subseteq \{-1, 0, 1\}^d \setminus \{0\}$ be a set of unit steps in dimension $d$. If $S$ is symmetric with respect to each axis, and $S$ takes a positive step in each direction, then the number of walks of length $n$ taking steps in $S$, beginning at the origin, and never leaving the positive orthant has asymptotic expansion

$$c_n \sim \left[\left(\frac{s^{(1)} \cdots s^{(d)}}{\pi^{d/2}|S|^{d/2}}\right) \cdot n^{-d/2} \cdot |S|^n\right].$$

where $s^{(k)}$ denotes the number of steps in $S$ which have $k$th coordinate 1.

Proof. We begin by verifying that each point $(w, t_w) \in V \cap T(\rho)$ satisfies the conditions of Theorem 59:

1. $(w, t_w)$ is a simple pole

   As $V$ is smooth, the point $(w, t_w)$ is a simple pole.

2. $(w, t_w)$ is isolated

   This is proven in Proposition 58.

3. $tH_t$ does not vanish at $(w, t_w)$

   This follows from $t_wH_t(w, t_w) = 1/(w_1 \cdots w_d) \neq 0$.

4. $(w, t_w)$ is non-degenerate

   Directly taking partial derivatives in Equation (5.10) implies

   $$\tilde{f}_{\theta_j \theta_k}(0) = \begin{cases} w_jw_k \frac{P_{y_jy_k}(w)P(w) - P_{y_j}(w)P_{y_k}(w)}{P(w)^2} & : j \neq k \\ P_{y_jy_j}(w)P(w) + w_jP_{y_j}(w)P(w) - P_{y_j}(w)^2 & : j = k \end{cases}.$$  

   Since $P_{y_j}(y) = (1 - y_j^{-2})P^{(j)}_1(y)$ we see that $P_{y_j}(w) = 0$. Furthermore, one can calculate that $P_{y_jy_k}(w) = 2P^{(j)}_1(w)$ and $P_{y_jy_k}(w) = 0$ for $j \neq k$, so that the Hessian of $\tilde{f}(w)(\theta)$ at $0$ is a diagonal matrix and

   $$H_w = \frac{2^n}{P(w)^d} P^{(1)}_1(w) \cdots P^{(d)}_1(w).$$

   The proof of Proposition 58 implies that $P^{(k)}_1(w) \neq 0$ for any $1 \leq k \leq d$, so each $(w, t_w)$ is non-degenerate.
Thus, we can apply Corollary 3.7 of Pemantle and Wilson [54] and sum the expansions (5.11) at each point in \( V \cap T(\rho) \) to obtain the asymptotic expansion

\[
c_n \sim |S|^n \sum_{w \in V \cap T(\rho)} \left( \sum_{l \geq l_w} C_w^l n^{-(d+l)/2} \right)
\]

for constants \( C_w^l \), where \( l_w \) is the degree to which \( G(y, t) \) vanishes near \( (w, t_w) \). Since \( G(y, t) = (1 + y_1) \cdots (1 + y_d) \) vanishes at all points of \( w \in V \cap T(\rho) \) except for \( \rho = (1, 1/|S|) \), the dominant term of (5.15) is determined only by the contribution of \( w = \rho \). Substituting the value for \( H_\rho \) given by Equation (5.14) into Equation (5.12) gives the desired asymptotic result.

We conclude with two examples which illustrate the various situations which may arise. The robustness of this method in dealing with step sets which are missing symmetries is discussed in the next, and final, chapter of this thesis.

**Example 61.** Consider the model in three dimensions restricted to the positive octant taking the eight steps

\[
S = \{(-1,0,\pm 1), (1,0,\pm 1), (0,1,\pm 1), (0,-1,\pm 1)\}.
\]

The kernel equation here is

\[
xyz(1 - tP(x, y, z))C(x, y, z, t) = xyz - ty(z^2 + 1)C(0, y, z) - tx(z^2 + 1)C(x, 0, z) - t(x^2y + y^2x + y + x)C(x, y, 0) + txC(x, 0, 0) + tyC(0, y, 0),
\]

with kernel

\[
P(x, y, z) = (x + y + \bar{x} + \overline{y})(z + \overline{z}).
\]

The generalized orbit sum method implies \( C(t) = \Delta F(x, y, z, t) \) where

\[
F(x, y, z, t) = \frac{(\bar{x} - x)(\overline{y} - y)(\overline{z} - z)}{\bar{x}\overline{y}\overline{z}(1 - txyzP(x, y, z))} \cdot \frac{1}{(1 - x)(1 - y)(1 - z)}
\]

\[
= \frac{(1 + x)(1 + y)(1 + z)}{1 - t(z^2 + 1)(x + y)(xy + 1)}.
\]
Next, we verify that the denominator $H(x, z, y, t)$ of $F(x, y, z, t)$ is smooth – i.e., that $H$ and its partial derivatives don’t vanish together at any point. This can be checked automatically by computing a Groebner Basis of the ideal generated by $H$ and its partial derivatives.

In Maple pseudo-code:

```maple
> H := 1 - t(z^2 + 1)(x + y)(xy + 1):
> GroebnerBasis([H, Hx, Hy, Hz, Ht], plex(t, x, y, z));

[1]
```

The critical points can also be computed:

```maple
> GroebnerBasis([H, tH - xHx, tH - yHy, tH - zHz], plex(t, x, y, z));

[z^2 - 1, y^2 - 1, x - y, 8t - y]
```

giving the finitely minimal critical point $\rho = (1, 1, 1, 1/8)$, where

$$T(\rho) \cap V = \{(1, 1, 1, 1/8), (1, 1, -1, 1/8), (-1, -1, 1, -1/8), (-1, -1, -1, -1/8)\}.$$ 

The value of $H_w$ can be calculated at each point to be $1/4$. For instance:

```maple
> f := logP(1) - log P(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) :
> subs(theta_1 = 0, theta_2 = 0, theta_3 = 0, det(Hessian(f, [theta_1, theta_2, theta_3])));

1/4
```

Equation (5.12) then gives the asymptotic result

$$c_n \sim 4\sqrt{2} \cdot \pi^{-3/2} \cdot n^{-3/2} \cdot 8^n.$$ 

Example 62. Consider the model in three dimensions restricted to the positive octant taking the twelve steps

$$S = \{(-1, 0, \pm1), (1, 0, \pm1), (0, 1, \pm1), (0, -1, \pm1), (\pm1, 1, 0), (1, \pm1, 0)\}.$$ 

Now, the generalized orbit sum method implies $C(t) = \Delta F(x, y, z, t)$ where

$$F(x, y, z, t) = \frac{(1 + x)(1 + y)(1 + z)}{1 - t(z^2 + 1)(x + y)(xy + 1) - tz(y^2 + 1)(x^2 + 1)}.$$
The denominator $H(x, z, y, t)$ of $F(x, y, z, t)$ can again be verified to be smooth, but the ideal encoding the critical point equations is no longer zero dimensional; i.e., there are an infinite number of solutions of the critical point equations. For instance, the calculation

$$H := 1 - t(z^2 + 1)(x + y)(xy + 1) - tz(y^2 + 1)(x^2 + 1);$$

$$I := \text{subs}(x = 1, y = -1, \left[H, tH_t - xH_x, tH_t - yH_y, tH_t - zH_z\right]);$$

$$\text{GröbnerBasis}(I, \text{plex}(t, x, y, z));$$

$$[1 - 4tz]$$

shows that any point $(1, -1, z, 1/4z)$ with $z \neq 0$ is a non-isolated critical point. However, none of these points are minimal – so Proposition 58 is not contradicted – since $|(1)(-1)(z)(1/4z)| = 1/4 > 1/|S|$. \(\triangleright\)
Chapter 6

Generalizations

We conclude by highlighting some work currently in progress, as well as possible future work on lattice walks in restricted regions.

6.1 Non Symmetric Step Sets in an Orthant

If the conjecture of Christol [24] that every D-finite globally bounded function is the diagonal of a rational function is true, one would expect the methods of Chapter 5 to apply to any D-finite walk model, not just those with symmetric step sets. In fact, it is known that all models in the quarter plane which take unit steps and have a finite group – i.e., the 23 models in Table 3.2 – can be written as diagonals of multivariate rational functions (the orbit sum method gives the rational functions explicitly in 19 cases, and the other four each have algebraic generating functions which can be represented as diagonals through Theorem 19). Thus, the only obstacle to verifying the asymptotic estimates guessed by Bostan and Kauers [10] and listed in Table 3.1 is the use of analytic combinatorics in several variables is the presence of a non-smooth singular variety.

6.1.1 ACSV in the General Case

We now outline how to apply the results of Analytic Combinatorics in Several Variables when the singular variety is allowed to have a more complicated geometry; such analysis has not been worked out for any singular variety, but the text of Pemantle and Wilson [55] contains very general results encompassing many situations one would expect to encounter.
Suppose again that we have a rational function \( F(y) = \frac{G(y,t)}{H(y,t)} \) and wish to determine the asymptotics of the diagonal sequence

\[
f_n := [t^n] \Delta F(y, t) = [y_1^n] \cdots [y_d^n] [t^n] F(y, t),
\]

using the geometry of the singular variety \( V = \{ z \in \mathbb{C}^{d+1} : H(z) = 0 \} \). As in the smooth case, the analysis comes from determining the critical points which determine the expansion of \( F \) along its diagonal, and the minimal points which are those determining dominant asymptotics. Minimal points are defined in the same manner as the smooth case: a point \( z \in \mathbb{C}^{d+1} \) is minimal if the intersection of \( T(z) = \{ w \in \mathbb{C}^{d+1} : |w_j| \leq |z_j|, j = 1, \ldots, d+1 \} \) with \( V \) is a subset of the boundary of \( T(z) \).

The critical points of \( V \) are defined to be the critical points of the Morse height function \( f(y) = \frac{1}{d+1} \left( \log |y_1| + \cdots + \log |y_{d+1}| \right) \) mapping from \( \mathbb{C}^{d+1} \) to \( \mathbb{C} \) (i.e., the points where the Jacobian of \( f \) has less than maximal rank). To find these points, one must compute a Whitney Stratification of \( V \), which is a decomposition of \( V \) into smooth manifolds subject to certain tangency conditions at their intersections. Each stratum in this decomposition can be represented as an algebraic variety of dimension at most \( d \), minus possibly some varieties of lower dimensions, and solving a generalization of the critical point equations (5.7) for each of these varieties gives the critical points. In the smooth case, the critical points have the same definition in terms of the Morse height function but the stratification is trivial as \( V \) itself is smooth.

If there are a finite number of minimal critical points, and at each \( V \) is the intersection of a finite number of smooth varieties whose normals are linearly independent, then Pemantle and Wilson give details on computing the dominant asymptotics of \( f_n \) and Raichev and Wilson [57] show additionally how to determine sub-dominant asymptotic terms. The difficulty in computing the asymptotics lies in computing the critical points through Whitney stratification, then using those points to determine the results of Cauchy residue integrals. The complexity of the steps in this method makes general results like Theorem 60 incredibly difficult. Even when dealing with the 23 finite group models in the quarter plane, difficulties arise which are not covered in Pemantle and Wilson [55] (for instance, when the critical points of \( V \) are all removable singularities). It seems like this will be a source of great future work for lattice path models, but there is still more work to do.
6.2 Walk Models in an Octant

Determining asymptotics using diagonals of rational functions is only possible when dealing with D-finite generating functions. In order to generalize the classification methods discussed in Chapter 3 to settings other than the halfspace or quarter plane, Bostan, Bousquet-Mélou, Kauers, and Melczer [13] have been carrying out a survey of lattice walks with unit steps in three dimensions which are restricted to the non-negative octant, beginning with models whose step sets have cardinality at most 6.

Given a model with step set $S$, we say that the model is one dimensional if there exists one coordinate such that the collection of walks with steps in $S$ restricted to stay non-negative only in this coordinate gives exactly the walks in the original model. Likewise, a model is two dimensional if there exists two coordinates such that all walks on $S$ staying non-negative in these two coordinates gives exactly the walks in the original model. As any one dimensional model is equivalent to one restricted to a halfspace, all such models have algebraic generating functions and we do not analyze them further. It is possible to determine how many distinct non-trivial models arise in two and three dimensions.

**Proposition 63** (Bostan et al. [13]). The generating function for the number of models having dimension two or three, no unused step, and counted up to permutations of the coordinates, is

$$
I(u) = 73 u^3 + 979 u^4 + 6425 u^5 + 28071 u^6 + 91372 u^7 + 234716 u^8 + 492168 u^9 \\
+ 860382 u^{10} + 1271488 u^{11} + 1603184 u^{12} + 1734396 u^{13} + 1614372 u^{14} \\
+ 1293402 u^{15} + 890395 u^{16} + 524638 u^{17} + 263008 u^{18} + 111251 u^{19} \\
+ 39256 u^{20} + 11390 u^{21} + 2676 u^{22} + 500 u^{23} + 73 u^{24} + 9 u^{25} + u^{26}.
$$

In particular, there are 35 548 distinct models of cardinality at most 6, compared to the $(\binom{26}{0}) + (\binom{26}{1}) + \cdots + (\binom{26}{6}) = 313 912$ total number of models. We let $O(x, y, z, t)$ denote the multivariate generating function marking the endpoint of a walk along with its length.

6.2.1 Two Dimensional Walks

Although the walks which are one dimensional can be classified through a careful analysis, we were unable to do this for the two dimensional models. Instead, we used integer programming to algorithmically check whether for each step set some two of the three conditions induced by the step set were sufficient to restrict to the octant; in fact 14 744 of the
35 548 distinct models taking at most 6 steps are two dimensional. Once it is determined which coordinate is redundant, it is automatic to determine a quarter plane model that the octant model is equivalent to. For instance, if a model taking steps in $S \subseteq \{\pm 1, 0\}^3$ has its $z$-coordinate being redundant then there is a set of steps $S' \subseteq \{\pm 1, 0\}^2$ such that $O(x, y, 1, t)$ is the generating function marking endpoint and length of the quarter plane model taking steps in $S'$.

Each of these 14 744 octant models is equivalent to one of 527 distinct quarter plane models. Although Section 3.3 outlines a classification for quarter plane models with unit steps, the step sets that arise through the reductions above can have multiple copies of the same step. The kernel method as presented in Section 3.3 can still be applied to walks with multiple steps, however there were complications including new models with zero orbit sum and models that had to be proven D-finite through computational methods (similar to Gessel’s walk). In addition, computational guessing led us to conjecture that all new quarter plane models with infinite group which arose were not D-finite. Details will be contained in a forthcoming manuscript, and the results are given in Figure 6.1.

**Figure 6.1:** Breakdown of results and conjectures for distinct 2D models arising in the analysis. D-finiteness is meant in all variables, and the numbers in brackets count the number of models of cardinality 3, 4, 5 and 6 for each class.
6.2.2 Three Dimensional Walks

For walks in three dimensions we generalize the notion of the group used in the quarter plane in a straightforward way by writing the characteristic polynomial

\[ P(x, y, z) = \sum_{(i,j,k) \in S} x^i y^j z^k \]

as

\[ P(x, y, z) = \overline{A}A_-(y, z) + A_0(y, z) + xA_+(y, z) \]
\[ = \overline{y}B_-(x, z) + B_0(x, z) + yB_+(x, z) \]
\[ = \overline{z}C_-(x, y) + C_0(x, y) + zC_+(x, y). \]

Any three dimensional model has \( A_+, B_+ \) and \( C_+ \) all non-zero, and we define the group of a model to be the group \( G \) of birational transformations of the variables \([x, y, z]\) generated by the following three involutions:

\[ \iota([x, y, z]) = \left[ \frac{\overline{A}A_-(y, z)}{A_+(y, z)}, y, z \right], \quad \psi([x, y, z]) = \left[ x, \frac{\overline{y}B_-(x, z)}{B_+(x, z)}, z \right], \]
\[ \tau([x, y, z]) = \left[ x, y, \frac{\overline{z}C_-(x, y)}{C_+(x, y)} \right]. \]

By construction, \( G \) fixes the Laurent polynomial \( P(x, y, z) \), and when \( G \) is finite we can use the kernel method to express the generating function as

\[ xyzO(x, y, z; t) = [x>0][y>0][z>0] \frac{1}{1 - tP(x, y, z)} \sum_{g \in G} \text{sgn}(g) g(xyz) \]

provided the rational function on the right hand side is non-zero, where \( \text{sgn}(g) \) is minus one to the length of a minimal word in \( \iota, \psi \) and \( \tau \) expressing \( g \).

We found 170 groups of finite order at most 48, and verified that all other groups had at least 200 elements. Of the finite group walks, 108 had a non-zero orbit sum and were thus proven D-finite. For 43 of the remaining walks with zero orbit sum, we were able to write the generating function \( O(x, y, z, t) \) as the Hadamard product of D-finite generating functions counting walks restricted to a halfspace or quarter plane, which implied that they were also D-finite. The remaining 19 walks with finite group but zero orbit sum are a mystery. Although they have a finite group, we are unable to prove they are D-finite and were unable to guess annihilating differential equations of \( O(x_0, y_0, z_0, t) \) for all combinations
6.3 Walks With Long Steps

In addition to studying walks in higher dimensions, another clear generalization is to study walks with longer (non unit) steps. We conclude this chapter with an example from ongoing work of Bostan, Bousquet-Méloü, and Melczer showing how a such a model can be analyzed.
Example 64. Consider the model with step set $\mathcal{S} = \{(0,1), (1, -1), (-2, -1)\}$ which is restricted to the non-negative quadrant. If we let $C(x, y, t)$ be the generating function marking endpoint and length, we can follow the kernel method as usual to obtain

$$K(x, y, t)C(x, y, t) = 1 - \frac{t(x^2 + 1)}{y^2 x^2}C(x, 0, t) - \frac{t}{x^2 y}C(0, y, t) + \frac{t}{x^2 y}C(0, 0, t),$$

(6.1)

where $C_{x=1}(y, t) := [x^1]C(x, y, t)$ and $K(x, y, t) = 1 - t \sum_{(i, j) \in \mathcal{S}} x^i y^j$. Solving the equations $K(a(x, y), y, t) = K(x, y, t)$ and $K(x, b(x, y), t) = K(x, y, t)$ for non-trivial solutions implies that $K(x, y, t)$ is fixed under the maps $\psi, \phi_1, \phi_2$ defined by

$$\psi : [x, y] \mapsto [x, \frac{x^3 + 1}{x^2 y}], \, \phi_1 : [x, y] \mapsto \left[1 + \sqrt{1 + 4x^3}, \frac{y}{2x^2}\right], \, \phi_2 : [x, y] \mapsto \left[1 - \sqrt{1 + 4x^3}, \frac{y}{2x^2}\right],$$

which are no longer rational maps but now live in the subring of Laurent series over $\mathbb{Q}$ which contain only a finite number of negative exponents. Composing these maps gives a group of maps of order six which preserve the kernel (see Aparicio-Monforte and Kauers [2] for details on the well defined composition of Laurent series), and applying each to the kernel equation (6.1) gives a set of six equations

$$C \left(1 - \sqrt{1 + 4x^3}, \frac{y}{2x^2}\right) = \frac{y}{2x^2} - V_1 - V_2 + V_3 \frac{1 - \sqrt{1 + 4x^3}}{2x}$$

$$C \left(1 - \sqrt{1 + 4x^3}, \frac{x^3 + 1}{x^2 y}, t\right) = \frac{(1 - \sqrt{1 + 4x^3})(x^3 + 1)}{2x^4 y} - V_1 - V_4 + V_5 \frac{1 - \sqrt{1 + 4x^3}}{2x}$$

$$\vdots$$

$$C \left(x, \frac{x^3 + 1}{x^2 y}, t\right) = \frac{x^3 + 1}{x^2 y} - V_7 - V_4 - V_5 \frac{1}{x}$$

in seven unknown Laurent series $V_1, \ldots, V_7$ representing the application of the group elements to the unknown functions on the right hand side of Equation (6.1). This, in turn, determines a homogeneous linear system of seven equations in six unknowns whose solution would allow one to find a linear combination of the identities in (6.2) eliminating $V_1, \ldots, V_7$. Although this system is under-determined there is a solution in this case, and taking such a linear combination followed by a non-negative power extraction in $x$ and $y$ implies

$$C(x, y, t) = [x^0][y^0] \frac{x^6 - x^5 y^2 - x^3 - 2x^2 y^2 - 2}{x^3 y(-x^2 y + t x^2 y^2 + t x^3 + t)}.$$
proving that $C(x, y, t)$ is D-finite. In our (rather limited) computations it appears that the resulting system can always be solved when the group above has order six, although we have encountered problems when dealing with groups of larger size (for instance size twelve).
Chapter 7

Conclusion

In this thesis we have examined variants of the kernel method, focusing on how the basic argument can be enhanced with results from different branches of mathematics. Although we deal with lattice path models here, these strategies for classifying generating functions and determining asymptotics have great applicability to many other combinatorial problems. In fact, if the conjecture of Christol [24] that every globally bounded D-finite function can be written as a rational diagonal is proven true in a constructive manner, the arguments in Chapter 5 illustrate how analytic combinatorics in several variables can be used to find the asymptotics of any globally bounded D-finite function (at least in the case of smooth singular varieties).

There are still many open questions in the study of lattice path models and D-finite functions, beginning with Christol’s conjecture and its weaker version for lattice path models.

**Question 1.** If the generating function of a lattice path model restricted to an orthant is D-finite, is it necessarily the diagonal of a multivariate rational function?

**Question 2.** Is every globally bounded D-finite function the diagonal of a multivariate rational function? [Conjectured by Christol [24]]

For lattice path models themselves, one large open question is to fully understand the interaction between the group of a model and the D-finiteness of its generating function. In many cases the finiteness of this group translates directly into a proof of D-finiteness for the related generating function, but when the orbit sum method gives a trivial result – i.e., the model has a zero orbit sum – this connection is not always clear.
Question 3. Do the 51 non-singular models with infinite group in Figure 3.3 admit non D-finite generating functions $C(t)$? [Conjectured by Mishna [51] and Bousquet-Mélou and Mishna [18], amongst others]

Question 4. For lattice path models restricted to the quarter plane taking non-unit steps, is it possible to define a group of transformations fixing the kernel $K(x, y, t)$ such that the group is finite if and only if the model admits a D-finite generating function?

Question 5. Are there families of lattice path models taking non-unit steps, or in higher than two dimensions, to which the iterated kernel method can be applied to prove non D-finiteness? [See Melczer and Mishna [50]]

In fact, as discussed in Section 6.2, there is some computational evidence that there exist models in three dimensions restricted to the positive octant which have finite groups but may admit non D-finite generating functions. This could imply that the correspondence which seems apparent in two dimensions may not hold in general.

Question 6. Are there lattice path models restricted to the positive octant taking unit steps which have a finite group $G$ (as defined in Section 6.2.2) and non D-finite generating function? If so, is it possible to re-define the group $G$ to eliminate this dysfunction? [See Bostan et al. [13]]

Finally, the work of Chapter 5 shows that step sets with a symmetry across each axis can be represented as the diagonal of a rational function whose denominator has a smooth variety. It is natural to wonder about the nature of this connection more generally.

Question 7. Suppose the generating function of a lattice path model in an orthant can be written as the diagonal of a rational function. How do the geometric properties of the singular variety connect to combinatorial properties of the model?

Question 8. Is there an automatic way to write the generating function of an algebraic (or D-finite) lattice path model with zero orbit sum as the diagonal of a rational function?
Bibliography


Definitions

combinatorial class, 6
counting sequence, 6
critical point, 62
equations, 62
isolated, 63

formal derivative, 6
formal Laurent series, 8
formal power series, 6

generating function, 6
algebraic, 9
D-Finite, 11
diagonal, 13
rational, 8
globally bounded, 16
group of a model, 26

Hadamard product, 17
holonomic sequence, 11

integer lattice path model, 18

kernel, 20
kernel equation, 20

Laurent polynomials, 8
linear recurrence
  w/ constant coefficients, 8
  w/ polynomial coefficients, 11

minimal point
  finite, 63
  strict, 63
minimal polynomial, 10

Padé-Hermite approximant, 31

singular quarter plane model, 38
singular variety, 61
step set of a walk, 18