MATH 152 — Quiz #2

March 13, 1997

Problem 1 *Differential equations.* A bursary fund starts with an initial endowment of \$50,000. It is expected that the fund will raise \$4,000 monthly. Each month the entire interest income plus 2% of the total fund is paid out in bursaries.

1. Denoting by $A(t)$ the amount of money in the fund, where time t is measured in months, find a differential equation for $A(t)$.
2. Find the solution of this differential equation with the appropriate initial condition.
3. How much money is in the bursary fund in the long run ($t \to \infty$)?

Solution of Problem 1.

1. Denote by $\alpha = 0.02$, $\beta = 4000$, and $A_0 = 50000$. The differential equation is

$$A'(t) = \beta - \alpha A(t) = 4000 - 0.02 A(t). \tag{1}$$

Note, that this is the same equation as the one for the pollution problem which we discussed in class. Most people had problems deriving it.

2. Seperating the variables, gives $\frac{dA}{\beta - \alpha A} = dt$, or

$$\begin{aligned} \frac{dA}{A - \frac{\beta}{\alpha}} = -\alpha dt, \quad &\rightarrow \quad \int \frac{dA}{A - \frac{\beta}{\alpha}} = \int -\alpha dt \\ \log |A - \frac{\beta}{\alpha}| = -\alpha t + c_1, \quad &\rightarrow \quad A - \frac{\beta}{\alpha} = \pm c_0 e^{-\alpha t}, \quad c_0 = e^{c_1} > 0 \\ A \quad &= \quad \frac{\beta}{\alpha} + c e^{-\alpha t}. \end{aligned}$$

Taking into account the initial condition $A(0) = A_0$, the solution is

$$A(t) = \frac{\beta}{\alpha} + \left(A_0 - \frac{\beta}{\alpha} \right) e^{-\alpha t}. \tag{2}$$

For our specific constants we have

$$A(t) = 200000 - 150000 e^{-0.02 t}. \tag{3}$$

3. From (3) we get

$$A(t) \rightarrow 200000;$$

this can also be seen from (1) directly, by setting $A'(t) = 0$, giving us $A = \beta/\alpha = 200000$. This is the equilibrium, where the incoming \$4000 exactly balance the outgoing 2% of the fund.

Problem 2 *Differential Equations.*

Find the solution y with $y(0) = 0$ of the linear first order equation

$$y'(x) + 2y(x) = 2x,$$

using variation of constants (or, by solving the homogeneous equation, and "guessing" a particular solution).

Solution of Problem 2.

Note, that in class we solved the problem $y'(x) + 3y(x) = x$. The homogeneous part of the equation, $y'(x) + 2y(x) = 0$, has the solution $y(x) = ce^{-2x}$. Variation of constants gives $y(x) = c(x)e^{-2x}$; plugging this into the equation, results into

$$c'(x)e^{-2x} - 2c(x)e^{-2x} + 2c(x)e^{-2x} = 2x,$$

i.e.,

$$c'(x) = 2xe^{2x}.$$

Integration by parts yields

$$c(x) = xe^{2x} - \int e^{2x}dx = xe^{2x} - \frac{1}{2}e^{2x} + k.$$

Thus, the general solution is

$$y(x) = c(x)e^{-2x} = x - \frac{1}{2} + ke^{-2x},$$

and the choice $k = \frac{1}{2}$ leads to $y(0) = 0$.

Alternatively, we can try to find a particular solution of the form $y(x) = \alpha x + \beta$; plugging into the differential equation

$$\alpha + 2(\alpha x + \beta) = 2x,$$

gives us $\alpha = 1$, $\beta = -\frac{1}{2}$, and thus $y = x - \frac{1}{2}$. Adding the general solution of the homogeneous equation leads to the same answer as above.

You may also follow the method in the book: Since $p(x) = 2$, we can multiply the equation by the integrating factor

$$\exp\left(\int 2dx\right) = e^{2x};$$

$$\begin{aligned} y'(x)e^{2x} + 2y(x)e^{2x} &= \frac{d}{dx}\left(y(x)e^{2x}\right) = 2xe^{2x} \\ y(x)e^{2x} &= \int 2xe^{2x}dx = \ldots = (x - \frac{1}{2})e^{2x} + k \\ y(x) &= x - \frac{1}{2} + ke^{-2x}. \end{aligned}$$

Problem 3 *Integration techniques.*

(a) What is

$$\int \sin^3 x \cos^2 x \; dx?$$

(b) Evaluate the integral

$$\int xe^{-2x} dx.$$

(c) What is

$$\int_0^{\pi^2} \frac{1}{2} \cos \sqrt{x}\, dx \quad ?$$

Solution of Problem 3.

(a) Substitute $\sin^2 x = 1 - \cos^2 x$, to obtain

$$\int \sin^3 x \cos^2 x \; dx = \int (\cos^2 x - \cos^4 x) \sin x \; dx;$$

furthermore, set $u = \cos x$, $du = -\sin x \; dx$, and the integral becomes

$$\int (u^4 - u^2) du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C.$$

(b) Partial integration gives

$$\int xe^{-2x} dx = -\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x} dx = (-\frac{1}{2}x - \frac{1}{4})e^{-2x} + C.$$

(c) Substitute $u = \sqrt{x}$, $u^2 = x$, $2udu = dx$, to obtain

$$\int_0^{\pi^2} \frac{1}{2} \cos \sqrt{x}\, dx = \int_0^{\pi} u \cos u \; du = [u \sin u]_0^{\pi} - \int_0^{\pi} \sin u \; du = 0 + [\cos u]_0^{\pi} = -2.$$

Problem 4 *Integration techniques.*

What is

$$\int \frac{x^4 + 1}{x(x^2 + 4)} dx \quad ?$$

Solution of Problem 4.

Polynomial division results in

$$\begin{aligned} \frac{x^4 + 1}{x(x^2 + 4)} &= x + \frac{1 - 4x^2}{x(x^2 + 4)}; \\ \frac{1 - 4x^2}{x(x^2 + 4)} &= \frac{A}{x} + \frac{Bx + C}{x^2 + 4} \\ 1 - 4x^2 &= A(x^2 + 4) + x(Bx + C) = (A + B)x^2 + Cx + 4A. \end{aligned}$$

Comparing the coefficients we find $A = \frac{1}{4}$, $\qquad C = 0$, $\qquad B = -\frac{17}{4}$, thus

$$\frac{x^4 + 1}{x(x^2 + 4)} = x + \frac{1}{4}\frac{1}{x} - \frac{17}{4}\frac{x}{x^2 + 4}.$$

This can be easily integrated:

$$\int \frac{x^4+1}{x(x^2+4)}dx = \frac{1}{2}x^2 + \frac{1}{4}\log|x| - \frac{17}{8}\log(x^2+4) + C.$$

Problem 5 *Integration techniques.* Compute

$$\int \frac{1}{\sin x + \tan x}dx.$$

Solution of Problem 5.

(Text, 7.5, Exercise 26.) This integral can be computed via the $u = \tan\frac{x}{2}$ substitution. The relevant formulas can be found by considering Figure 1.

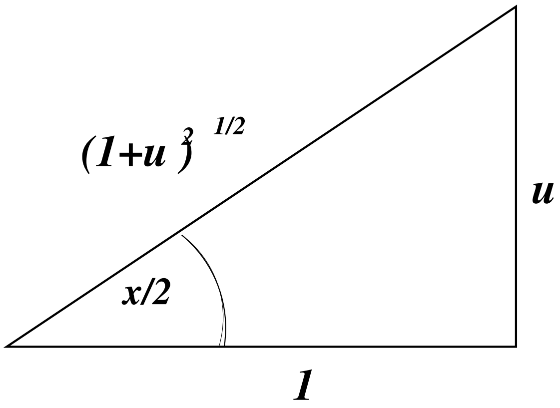

Figure 1: $\tan\frac{x}{2}$ Triangle

$$\begin{aligned} u &= \tan\frac{x}{2} & \sin x &= 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2u}{1+u^2} \\ \frac{du}{dx} = \frac{1/2}{\cos^2\frac{x}{2}} &= \frac{1}{2}(1+u^2) & \cos x &= \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{1-u^2}{1+u^2} \\ & & \tan x &= \frac{\sin x}{\cos x} = \frac{2u}{1-u^2}. \end{aligned}$$

$$\begin{aligned} \int \frac{1}{\sin x + \tan x}dx &= \int \frac{1}{\frac{2u}{1+u^2} + \frac{2u}{1-u^2}} \frac{2}{1+u^2}\,du = \int \frac{1}{u\left(1 + \frac{1+u^2}{1-u^2}\right)}du \\ &= \int \frac{1}{u\left(\frac{1-u^2+1+u^2}{1-u^2}\right)}du = \frac{1}{2}\int \frac{1-u^2}{u}\,du \\ &= \frac{1}{2}\int \frac{du}{u} - \frac{1}{2}\int u du = \frac{1}{2}\log|u| - \frac{1}{4}u^2 + C \\ &= \frac{1}{2}\log\left|\tan\frac{x}{2}\right| - \frac{1}{4}\tan^2\frac{x}{2} + C. \end{aligned}$$