MATH 929 — PROJECT 3

Report by Tuesday, April 01, 1997

Deadline for Project 2 is also April 1 — no April Fool's joke.

Problem 1 Prove Lemma 3.2 on page 30 of the class notes.

Problem 2 Write programs to solve the tridiagonal symmetric eigenvalue problem based on the Sturm sequence (see class notes, § 3). Compare the accuracy to the built-in MATLAB solver.

We investigate the two problems

$$u_{tt} = u_{xx}, \qquad \text{on } [0,1], \qquad u(0) = u(1) = 0, \tag{3.1}$$

the vibrating string, and the 2D wave equation

$$u_{tt} = \Delta u = u_{xx} + u_{yy}, \tag{3.2}$$

describing the movements of a membrane stretched over a square frame.

We use the finite difference approximations described in Project 2. For the 2-D case we let n be a positive integer, set $h = 1/n$, and consider an approximation

$$u_{ij} \approx u(ih, jh), \qquad 0 \leq i, j \leq n$$

to $u(x, y)$. The Laplacian operator on the right in (3.2) will be discretized by

$$(\Delta_h u)_{ij} = -\frac{1}{h^2}\left(4u_{ij} - u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1}\right).$$

Assuming that u_{ij} is specified on the boundaries, there are $m = (n-1)^2$ unknown values in the interior. We can put those into an $m-$vector as follows:

$$v = (u_{11}, u_{12}, \ldots, u_{1,n-1}, u_{21}, \ldots, u_{2,n-1}, \ldots, u_{n-1,n-1})^T.$$

Problem 3 Our first goal is to compute the lowest frequency of the membrane. Mathematically this amounts to finding the smallest ω for which (3.2) has a solution of the form

$$u(x, y, t) = e^{i\omega t} u(x, y)$$

for some $u(x, y)$, i.e.,

$$-\omega^2 u = \Delta u, \qquad u = 0 \quad \text{on the boundary.} \tag{3.3}$$

(a) For this simple problem the lowest eigensolution is a product of sine waves in x and y. Plug in the appropriate function, and determine what the corresponding exact number ω^2 is.

(b) We discretize (3.3) to obtain $Lv = \omega^2 v$, where L is the $m \times m$ symmetric matrix corresponding to $-\Delta_h$. Write down what this matrix is. Make sure you get this right, as the rest of the problem set depends on it. Explain why L is nonnegative definite.

(c) Write a subroutine/function w=LMUL(v) which, given $v \in \mathbb{R}^m$, returns $w = Lv \in \mathbb{R}^m$. After checking LMUL carefully, write a program to execute the pure (no re-orthogonalization) Lanczos process described in the class notes. Taking $n = 31$, $m = 900$, and beginning with the initial vector $q_1 = (1, \ldots, 1)^T/(n-1)$, and with a random q_1, carry out 40 steps of the Lanczos iteration, to obtain the resulting tridiagonal matrix T_{40}.

(d) Compute the lowest eigenvalue of T_j for $j = 5, 10, 15, 20, 25, 30, 35, 40$. How fast does the convergence appear to be? What is the best estimate for the exact eigenvalue? Compare to your answer of (a)?

(e) Try to find approximations to the first few of the lower eigenvalues.

Problem 4 Study the one-dimensional problem more carefully. Try to find the eigenvalues of the discretized operator A, and apply your methods to finding the eigenvalues of A^{-1}. Do not form the matrix A^{-1}, rather find a way to efficiently implement the product $u = A^{-1}x$, i.e., given x find u such that $Au = x$. What are your observations, conclusions?