MATH 155 — Lecture #6

January 19, 2000

Trigonometric Substitution. In this section we learn how to find antiderivatives of functions which contain expressions of the form $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, and $\sqrt{x^2-a^2}$.

(a) $\int \sqrt{1-x^2}dx =$? We use the substitution $x = \sin\theta$. Note that $\sin\theta$ maps the interval $[-\pi/2, \pi/2]$ on the interval $[-1, +1]$, which is the domain of definition of our function of interest, $\sqrt{1-x^2}$. This substitution succeeds, because

$$\cos^2\theta + \sin^2\theta \equiv 1 \qquad \Rightarrow \qquad \sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta.$$

Note, that for $-\pi/2 \leq \theta \leq +\pi/2$ we have $\cos\theta \geq 0$, so we have the correct square root. To summarize, here are all the transformation formulas:

$$x = \sin\theta, \qquad \frac{dx}{d\theta} = \cos\theta \;\Rightarrow\; dx = \cos\theta d\theta, \qquad \sqrt{1-x^2} = \cos\theta.$$

After substituting we obtain

$$\int \sqrt{1-x^2}dx = \int \cos\theta\cos\theta d\theta = \int \cos^2\theta d\theta.$$

To evaluate this integral we use the trigonometric identities

$$\cos^2\theta = \frac{1}{2}\left(1+\cos(2\theta)\right), \qquad \sin(2\theta) = 2\cos\theta\sin\theta.$$

$$\begin{aligned}\int \sqrt{1-x^2}dx &= \int \cos^2\theta d\theta = \int \frac{1}{2}\left(1+\cos(2\theta)\right)d\theta = \frac{1}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right) + C \\ &= \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C = \frac{1}{2}\theta + \frac{1}{2}\cos\theta\sin\theta + C = \frac{1}{2}\sin^{-1}(x) + \frac{1}{2}x\sqrt{1-x^2} + C.\end{aligned}$$

(b)

$$\int \frac{\sqrt{4-x^2}}{x^2}dx = \int \frac{2\cos\theta}{4\sin^2\theta}2\cos\theta d\theta = \int \cot^2\theta d\theta,$$

where we have used the substitution

$$x = 2\sin\theta, \qquad dx = 2\cos\theta d\theta, \qquad \sqrt{4-x^2} = 2\cos\theta.$$

To find the antiderivative of $\cot^2\theta$ we may proceed as follows:

$$\begin{aligned}\int \cot^2\theta d\theta &= \int \frac{\cos^2\theta}{\sin^2\theta}d\theta = \int \frac{1-\sin^2\theta}{\sin^2\theta}d\theta \\ &= \int \frac{1}{\sin^2\theta}d\theta - \int 1 d\theta = -\cot\theta - \theta + C \\ &= -\frac{1}{x}\sqrt{4-x^2} - \sin^{-1}\left(\frac{x}{2}\right) + C.\end{aligned}$$

(c) For integrals involving a term

$$\sqrt{x^2+a^2}$$

we use the substitution

$$x = a\tan\theta, \qquad x^2+a^2 = a^2(\tan^2\theta+1) = a^2\sec^2\theta \;\Rightarrow\; \sqrt{x^2+a^2} = a\sec\theta, \qquad dx = a\sec^2\theta d\theta.$$