MATH 155 — Lecture #7

January 21, 2000 (Lecture was cancelled)

Trigonometric Substitution - more examples. Find the antiderivative of

$$\frac{1}{\sqrt{x^2-a^2}}, \qquad (a>0).$$

We need $|x| \geq a$ for the square root to be defined.

We use the substitution

$$x = a\sec\theta = a\frac{1}{\cos\theta}.$$

The function $a\sec\theta$ maps the interval $[0,\pi/2)$ onto the interval $[a,\infty)$.

$$\sec^2\theta - 1 = \frac{1}{\cos^2\theta} - 1 = \frac{1-\cos^2\theta}{\cos^2\theta} = \frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta \qquad \Rightarrow \qquad \sqrt{x^2-1} = \tan\theta \qquad (0 \leq \theta < \pi/2).$$

To summarize, here are all the transformation formulas:

$$x = a\sec\theta, \qquad \frac{dx}{d\theta} = -\frac{-a\sin\theta}{\cos^2\theta} = a\tan\theta\sec\theta, \qquad \sqrt{x^2-a^2} = a\tan\theta.$$

After substituting we obtain

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \int \frac{1}{a\tan\theta}a\tan\theta\sec\theta d\theta = \int \sec\theta d\theta = \int \frac{1}{\cos\theta}d\theta.$$

It is not entirely straightforward to evaluate this integral, but you can check the derivatives:

$$\int \frac{1}{\cos\theta}d\theta = \ln\left(\frac{1+\sin\theta}{\cos\theta}\right) + C = \ln\left(\sec\theta + \tan\theta\right) + C.$$

Backsubstitute, and you obtain

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \ln\left(\frac{x}{a} + \frac{1}{a}\sqrt{x^2-a^2}\right) + C \underbrace{=}_{why?} \ln\left(x + \sqrt{x^2-a^2}\right) + \tilde{C}. \tag{1}$$

Note: This integral can also be attacked using hyperbolic functions. If you have not heard of those, you can ignore them. Briefly,

$$\cosh t = \frac{e^t+e^{-t}}{2}, \qquad \sinh t = \frac{e^t-e^{-t}}{2}, \qquad \tanh t = \frac{\sinh t}{\cosh t} = \frac{e^t-e^{-t}}{e^t+e^{-t}}.$$

Some interesting formulas are

$$\cosh^2 t - \sinh^2 t = 1, \qquad \frac{d}{dt}\cosh t = \sinh t, \qquad \frac{d}{dt}\sinh t = \cosh t.$$

You can then substitute $x = a\cosh t$, so $\sqrt{x^2-a^2} = a\sinh t$. Furthermore, $dx = a\sinh t dt$:

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \int 1 dt = t + C.$$

Backsubstitute to confirm (1), the previous result (left as an exercise: not difficult, but not trivial either).

Footote: Hyperbolic and trigonometric functions are closely related. You can see the similarity when moving to complex numbers. Let $i = \sqrt{-1}$. Then, for real t, we have

$$e^{it} = \cos t + i\sin t, \qquad \cos t = \frac{e^{it}+e^{-it}}{2}, \qquad \sin t = \frac{e^{it}-e^{-it}}{2}.$$