MATH 155 — Lecture #9

January 26, 2000

Sums . Notation. We use the greek $\sum$ symbol to express summation.

$$\sum_{k=1}^{n} x_k = x_1 + x_2 + x_3 + \cdots + x_{n-1} + x_n = \sum_{j=1}^{n} x_j.$$

Note that the summation index (k or j) can have any name, not changing the value of the sum. You can think of this like a simple computer program:

```
s := 0;
for k = 1 to n do
    s := s + x[k];
end if
```

The empty sum equals zero, e.g. $\sum_{k=1}^{0} x_k = 0$. In the next section we will encounter sums of the form

$$\sum_{k=1}^{n} f(x_k) = f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_{n-1}) + f(x_n).$$

Values for some sums we will be using in the next few sections:

$$\begin{aligned}
\sum_{k=1}^{n} 1 &= n, \\
\sum_{k=1}^{n} k &= \frac{n(n+1)}{2} &= \frac{n^2}{2} + O(n), \\
\sum_{k=1}^{n} k^2 &= \frac{n(n+1)(2n+1)}{6} &= \frac{n^3}{3} + O(n^2), \\
\sum_{k=1}^{n} k^3 &= \frac{n^2(n+1)^2}{4} &= \frac{n^4}{4} + O(n^3).
\end{aligned}$$

The symbol $O(n^p)$ describes a term which is bounded by a constant times n^p.

In the lecture we found the formula for $\sum_{k=1}^{n} k$. Here we demonstrate how to find the formula for $\sum_{k=1}^{n} k^3$ using telescoping sums, and the formulas for $\sum_{k=1}^{n} k^2$, $\sum_{k=1}^{n} k$, and $\sum_{k=1}^{n} 1$.

Set $S = \sum_{k=1}^{n} k^3$. We have

$$\begin{aligned}
(k+1)^4 - k^4 &= 4k^3 + 6k^2 + 4k + 1 \\
\sum_{k=1}^{n} [(k+1)^4 - k^4] &= \sum_{k=1}^{n} [4k^3 + 6k^2 + 4k + 1].
\end{aligned}$$

The left hand side is a telescoping sum

$$\sum_{k=1}^{n} [(k+1)^4 - k^4] = (n+1)^4 - 1 = n^4 + 4n^3 + 6n^2 + 4n.$$

The right hand side is

$$\begin{aligned}\sum_{k=1}^{n}[4k^3+6k^2+4k+1] &= 4\sum_{k=1}^{n}k^3+6\sum_{k=1}^{n}k^2+4\sum_{k=1}^{n}k+\sum_{k=1}^{n}1\\ &= 4S+n(n+1)(2n+1)+2n(n+1)+n\\ &= 4S+n(n+1)(2n+3)+n.\end{aligned}$$

Therefore,

$$\begin{aligned}4S &= n^4+4n^3+6n^2+4n-n\underbrace{(n+1)(2n+3)}_{(2n^2+5n+3)}-n\\ &= n^4+4n^3+6n^2+4n-2n^3-5n^2-3n-n\\ &= n^4+2n^3+n^2=n^2(n^2+2n+1)=n^2(n+1)^2.\end{aligned}$$

Another important sum comes from the geometric series

$$s=\sum_{k=0}^{n}q^k=1+q+q^2+\cdots+q^{n-1}+q^n.$$

To evaluate this series use the following trick:

$$\begin{array}{rclcccccccccccc} s &=& 1 &+& q &+& q^2 &+& \cdots &+& q^{n-1} &+& q^n & & \\ qs &=& & & q &+& q^2 &+& \cdots &+& q^{n-1} &+& q^n &+& q^{n+1} \\ \hline s-qs &=& 1 & & & & & & & & & & &-& q^{n+1} \end{array}$$

hence

$$s=\sum_{k=0}^{n}q^k=\frac{1-q^{n+1}}{1-q}=\frac{q^{n+1}-1}{q-1}.$$