MATH 155 — Lecture #20–22

February 28 – March 3, 2000

4 Differential Equations

Many natural, scientific and economic phenomena can be described in terms of **differential equations**, or stated in less technical language, in terms of **rates of changes**. These notes include some examples presented in class, and also some extra material (particularly Section 4.2).

Here we will only discuss **ordinary** differential equations (ODEs); these are equations in which only derivatives with respect to one variable occur. **Partial** differential equations (PDEs) are relations between partial derivatives with respect to two or more different variables.

Before discussing some theoretical background and a simple numerical method, we give a number of examples.

4.1 ODEs - Introductory examples

Our first example refers to integration of a function.

Example 4.1 (Quadrature) *The problem of finding the antiderivative of a function f can be phrased as a differential equation, namely*

$$\frac{dy}{dx} = y'(x) = f(x). \tag{4.1}$$

From elementary calculus we know that that this cannot always be solved in closed form, i.e., if $f(x)$ is given by a formula in terms of elementary transcendental functions, we may not be able to find such a formula for $y(x)$. Even if this is possible, a number of sophisticated tricks may have to be employed (remember those calculus exams?). Furthermore, the solution of (4.1) is not unique, as we may add a constant, i.e., with $y(x)$ every function $y(x)+c$ is also a solution. To make the solution unique, we can specify the value of the solution at a given point, for example $y(0) = y_0$. With this condition the solution of (4.1) can be written as

$$y(x) = y_0 + \int_0^x f(x)dx, \tag{4.2}$$

the fundamental theorem of calculus. As far as existence of a solution is concerned, we know that continuity of $f(x)$ is a sufficient condition. For completeness we mention that

$$\int_a^b f(x)dx = y(b) - y(a),$$

for any solution y of (4.1).

This example exhibits some of the essential features of differential equations, foremost the difficulty of finding solutions in closed form, and the fact that additional conditions (initial conditions or boundary conditions) are required to make the solution unique. The next example describes exponential growth or decay, and gives us a first glimpse of numerical methods.

Example 4.2 (Compound interest) *We investigate the effects of different compounding intervals on the final amount of money in a bank account after one year. Let us assume we start with an initial balance of \$10,000 ($A(0) = A_0 = 10000$), and the quoted annual rate of interest is 12% ($q = 0.12$). We denote by $A(t)$ the balance at time t, where t is measured in units of one year. If the interest is compounded annually ($m = 1$), then it is easy to see, that the balance after year 1 is $A(1) = A_0(1+q)$ (=\$11,200). Note, that the amount at time t is simply given by $A(t) = A_0(1+qt)$, $0 \le t \le 1$. Semiannual compounding ($m = 2$) gives*

$$A(1) = A_0(1+q/2)(1+q/2) = A_0(1+q+q^2/4)$$

(=\$11,236). Throughout the year, the balance in the account is given by

$$A(t) = \begin{cases} A_0(1+qt/2), & 0 \le t \le 1/2, \\ A_0(1+q/2)(1+q(t-1/2)/2), & 1/2 \le t \le 1. \end{cases}$$

More systematically, if we have m compounding periods within one year, it is still very easy to write down what happens within one compounding period, namely

$$A\left(\frac{k+1}{m}\right) = A\left(\frac{k}{m}\right)\left(1+\frac{q}{m}\right), \qquad k = 0,\dots,m-1, \tag{4.3}$$

and $A(t)$ is linear during a compounding interval. This leads immediately to the formula

$$A(1) = A_0\left(1+\frac{q}{m}\right)^m \tag{4.4}$$

for the case of m compounding periods. From calculus we know that letting $m \to \infty$, we obtain

$$A(1) = A_0 e^q \tag{4.5}$$

for continuous compounding.

So where is the differential equation? Rewriting (4.3) with $t = \frac{k}{m}$ and $\Delta t = \frac{1}{m}$, we obtain

$$\begin{aligned} A(t+\Delta t) &= A(t)(1+q\Delta t) \\ \frac{A(t+\Delta t) - A(t)}{\Delta t} &= qA(t), \end{aligned}$$

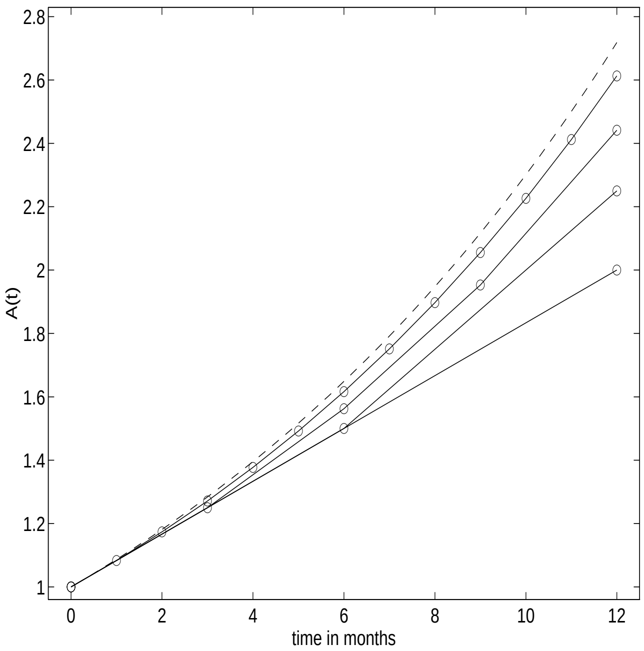

Figure 4.1: Discrete vs. continuous compounding with $q = 1$ (100% interest), or exact solution and numerical approximation (Euler's method) for Equation (4.6)

and finally, letting $\Delta t \to 0$, *we get*

$$A'(t) = qA(t). \tag{4.6}$$

This equation can be solved by what is known as the method of separation. *We will use the initial condition* $A(0) = A_0$, *and proceed as follows:*

$$\begin{aligned}
\frac{dA}{dt} &= qA \\
\frac{dA}{A} &= qdt \\
\int_{A_0}^{A} \frac{dA}{A} &= \int_0^t qdt \\
\log\left|\frac{A(t)}{A_0}\right| &= qt \\
\left|\frac{A(t)}{A_0}\right| &= e^{qt}
\end{aligned}$$

$$A(t) \quad = \quad A_0 e^{qt}$$

It is easy to check that the function $A(t) = A_0 e^{qt}$ indeed satisfies the differential equation (4.6) and the initial condition $A(0) = A_0$. *Of course, (4.5) is just the function $A(t)$ evaluated at $t = 1$. One could think of discrete compounding as an approximation to continuous compounding, see Figure 4.1 (a larger value of q had to be chosen to make the differences visible in the graph). For the balances in our bank account after one year we obtain the following values:*

Compounding	m	*Balance* $A(1)$
annual	*1*	*11200.00*
semi-annual	*2*	*11236.00*
quarterly	*4*	*11255.09*
monthly	*12*	*11268.25*
daily	*365*	*11274.75*
hourly	*8760*	*11274.96*
continuous	∞	*11274.97*

The differential equation (4.6) discussed in Example 4.2 describes exponential growth or decay (depending on the sign of q). Even though it can always be solved exactly (the exponential function is the solution), it serves as an important model problem for numerical methods as well. It is a first-order equation (only a first derivative appears), and it is a linear equation (the unknown function and its derivatives appear only linearly). Next we provide a simple example of a nonlinear differential equation.

Example 4.3 (Logistic growth equation) *Let $y(t)$ denote the ("approximate") number of individuals in a certain population (we are therefore restricting our attention to $y \geq 0$ for both equations (4.7) and (4.8)). If this population has a constant growth rate α (the difference between a constant birth rate and death rate), then the differential equation*

$$y'(t) = \alpha y(t), \tag{4.7}$$

with initial condition $y(0) = y_0$ describes the population growth. Clearly, for $\alpha < 0$ the population will become extinct (in this model in an infinite amount of time, although, for example, $y(t) = .47$ may not exactly indicate a lively bunch of representatives of that species; in fact, these kind of models make only sense for relatively large populations!). For $\alpha = 0$ the population remains constant, and for $\alpha > 0$ it exhibits exponential growth. Exponential growth may be happening in practice over short periods of time, but is not sustainable over the long run (a somewhat contentious statement, challenged by some of today's economists and religious leaders). To reflect other effects such as overcrowding, competition for food, etc., one could introduce a new term into the equation,

$$y'(t) = \alpha y(t) - \beta y(t)^2, \tag{4.8}$$

$\beta > 0$, but typically a small constant. Now the instant growth rates vary with the population count. In effect we have replaced the constant α in (4.7) by the term $\alpha - \beta y(t)$, which introduces a nonlinear term into the equation. In general, nonlinear equations are even more difficult to solve, although often it is possible to make a few general statements. First we could ask, whether there is an equilibrium *or* steady state, *i.e., a population level y_e for which there is no change, i.e., $y' = 0$. For (4.8) this means that y_e must be a solution of*

$$ay_e - \beta y_e^2 = 0.$$

One solution is the trivial solution $y_e = 0$, which is not very interesting. The second solution is the equilibrium

$$y_e = \frac{\alpha}{\beta}.$$

Without knowing the solution, just by looking at the differential equation, we observe that

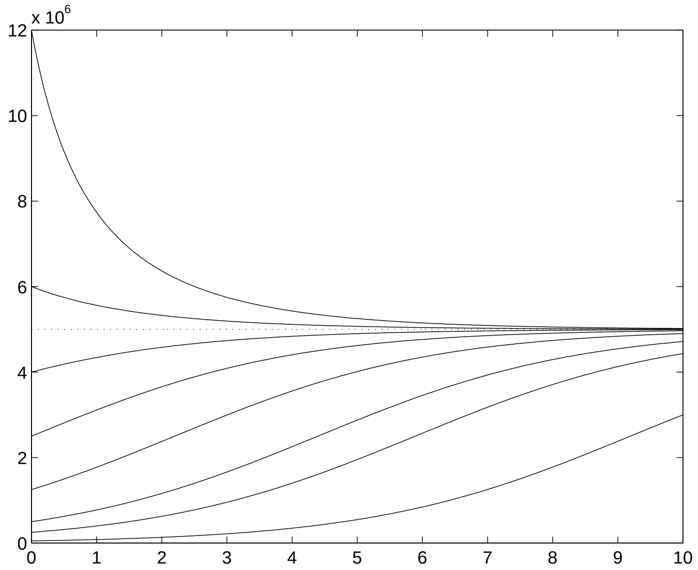

Figure 4.2: Solutions of logistic equation (4.8) with $\alpha = 0.5$ and $\beta = 10^{-7}$ for various initial values y_0

$y' > 0$ for $0 < y < y_e$, i.e., the solutions will grow for these y-values, and $y' < 0$ for $y > y_e$. This leads us to the concept of direction fields (see also Figure 4.2): The differential equation gives a certain direction at any point (t, y) in the (t, y)-plane, and all the solution curves must be tangential to these given directions. Our differential equation (4.8) can be solved in closed form, by the method of separation:

$$\frac{dy}{dt} = \alpha y - \beta y^2$$

$$\begin{aligned}
\frac{dy}{\alpha y - \beta y^2} &= dt \\
\frac{dy}{y - \frac{\beta}{\alpha} y^2} &= \alpha dt \\
\frac{\frac{\alpha}{\beta}}{y(\frac{\alpha}{\beta} - y)} dy &= \alpha dt \\
\left(\frac{1}{\frac{\alpha}{\beta} - y} + \frac{1}{y} \right) dy &= \alpha dt \\
\int \frac{dy}{\frac{\alpha}{\beta} - y} + \int \frac{dy}{y} &= \int \alpha dt \\
-\log |\frac{\alpha}{\beta} - y| + \log |y| &= \alpha t + \text{const} \\
\frac{y}{|\frac{\alpha}{\beta} - y|} &= C e^{\alpha t}
\end{aligned}$$

To find a formula for the solution y we have to distinguish between the cases $0 < y < y_e = \alpha/\beta$ and $y_e < y$. As the final result we obtain (with C a positive constant)

$$y(t) = \frac{\frac{\alpha}{\beta} C e^{\alpha t}}{1 + C e^{\alpha t}}, \qquad C = \frac{y_0}{|\frac{\alpha}{\beta} - y_0|}. \tag{4.9}$$

Figure 4.2 has a graph of the solution curves.

Next let us consider an example leading to a higher order differential equation. In physics, Newton's law of motion, stating that

$$\text{force} = \text{mass} \times \text{acceleration}$$

gives rise to second order differential equations; if $y(t)$ denotes the position of a particle, the $y'(t)$ is its velocity and $y''(t)$ is its acceleration. Our example is

Example 4.4 (Frictionless Pendulum) *To derive the equation we look at Figure 4.3. The pendulum has length ℓ, mass m. Its movement can be described by the angle ϕ, the deviation from its (stable) equilibrium position. The velocity and the acceleration are proportional to ℓ, and the force acting on the pendulum is $F = -mg \sin\phi$ where g is the gravitational constant ($g \approx 9.81ms^{-2}$). Thus Newton's law of motion states,*

$$m\ell\phi''(t) = -mg \sin\phi(t),$$

or

$$\phi''(t) + \omega^2 \sin\phi(t) = 0, \tag{4.10}$$

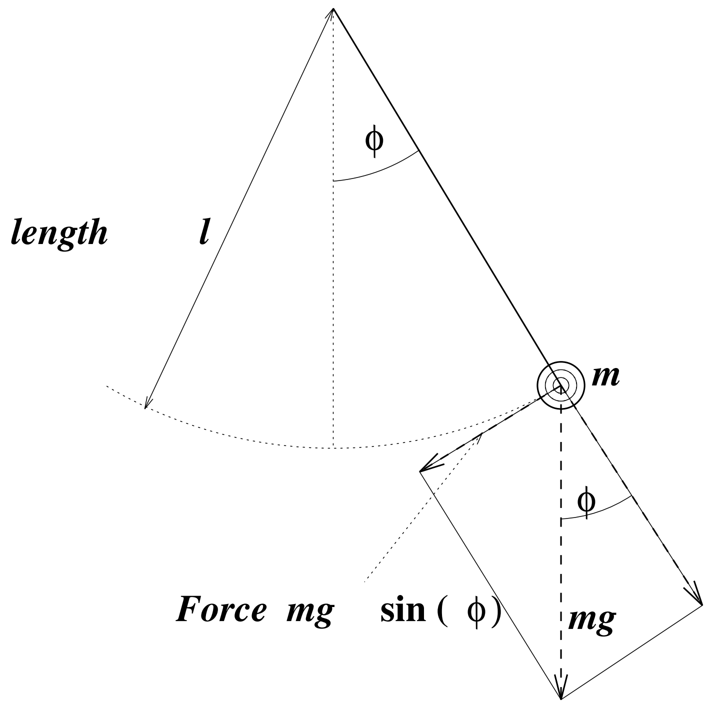

Figure 4.3: Pendulum

where $\omega^2 := \frac{g}{\ell}$*. This equation can still be solved exactly, but no longer in an elementary way (the exact solution involves so-called* elliptic functions*), but finding numerical approximations to the solution is not difficult. We can find approximate solutions to (4.10) by* **linearizing** *the equation around its equilibrium solution* $\phi \equiv 0$*. For small angles* ϕ *we have* $\sin\phi \approx \phi$*, and the linearized equation simply becomes*

$$\phi''(t) + \omega^2\phi(t) = 0. \tag{4.11}$$

It is easy to check that the function

$$\phi(t) = \alpha\cos(\omega t) + \beta\sin(\omega t)$$

is a solution of (4.11), which now contains two unknown constants. These need to be determined by initial conditions

$$\phi(0) = \phi_0, \qquad \phi'(0) = v_0.$$

It is obvious from physics, that we need to know the position and the velocity at time $t = 0$ *to determine the solution* ϕ*, and this is what is required mathematically as well. We note that the solution of the linearized problem is a periodic function with period* $T = \frac{2\pi}{\omega}$*. The* *solution of the nonlinear equation (4.10) is also periodic, but has a slightly different form and a different period (which depends on the amplitude* $\max|\phi(t)|$*). In the absence of friction, the pendulum will continue to swing with the same amplitude and frequency. The "grandfather clock" uses the principle of constant frequency to measure time; an almost frictionless state is achieved by taking a large mass* m*, and the speed of the clock can be adjusted by changing the length* ℓ *(larger* ℓ *means slower frequency). Changing* g *would be a lot harder, unless you are willing to relocate the clock into outer space.*

We finish this section with the remark, that a higher order differential equation like (4.10) can be rewritten as a first order **system** of differential equations. For (4.10) one proceeds as follows. Set

$$y_1 := \phi, \qquad y_2 := \phi'.$$

This gives the two equations

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -\omega^2 \sin(y_1), \end{aligned}$$

where the first one follows directly from the definition of y_1 and y_2, and the second one is equivalent to the original differential equation. With the vector $\boldsymbol{y}$ and vector function $\boldsymbol{f}$

$$\boldsymbol{y} := \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \boldsymbol{f}(t, \boldsymbol{y}) := \begin{pmatrix} y_2 \\ -\omega^2 \sin(y_1) \end{pmatrix}$$

the second order differential equation (4.10) can now be written as the first order system

$$\boldsymbol{y}' = \boldsymbol{f}(t, \boldsymbol{y}). \tag{4.12}$$

Although this conversion is trivial, it is important, as it allows us to restrict our attention completely to first order equations, both theoretically and numerically. Virtually all software for solving ordinary differential equations requires the equation to be converted to a first order system.

4.2 Existence and Uniqueness of Solutions

An ordinary differential equation is of the form

$$y'(t) = f(t, y(t)), \tag{4.13}$$

where $t \in \mathbb{R}$ is the time variable, and y and f are functions taking values in $\mathbb{R}$ or $\mathbb{C}$. We will frequently write (4.13) simply as $y_t = f$. To turn (4.13) into an initial value problem (IVP), we specify the initial condition

$$y(0) = y_0, \tag{4.14}$$

and seek solutions on some interval $[0, T]$, $T > 0$ (singling out the point $t = 0$ as the point where the initial condition is given is done without loss of generality). If the unknown function is a vector function, or after converting a higher order ODE to a first order system, we have the initial value problem

$$\boldsymbol{y}'(t) = \boldsymbol{f}(t, \boldsymbol{y}(t)), \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0. \tag{4.15}$$

We will not treat boundary value problems, in which components of $\boldsymbol{y}$ are specified at two or more points in time.

A differential equation is called **autonomous**, if the function f does not depend on t explicitly:

$$\boldsymbol{y}'(t) = \boldsymbol{f}(\boldsymbol{y}(t)), \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0. \tag{4.16}$$

Some software requires the differential equation to be in autonomous form. This can always be achieved, by introducing t as a new dependent variable, and adding one extra equation to the system.

To compute a solution we must be assured that a solution exists and that it is unique (or, at least, locally unique). For initial value problems there is a fairly simple theorem guaranteeing existence and uniqueness **locally**, i.e. for small values of t.

Theorem 4.5 (Picard's Theorem) *Assume the function $f(t, y)$ in (4.13) is continuous in a rectangle*

$$G = \{(t, y) : |t| \leq \bar{t}, |y - y_0| \leq \bar{y}\} \subset \mathbb{R}^2,$$

and satisfies a Lipschitz condition with respect to the second variable y on the rectangle G, i.e., there is a constant L, such that

$$|f(t, y_2) - f(t, y_1)| \leq L|y_2 - y_1|,$$

for (t, y_1) and $(t, y_2) \in G$. Then the initial value problem (4.13) and (4.14) has a unique solution $y(t)$ defined in a neighbourhood of the point $(0, y_0)$.

A few remarks are in order. The Lipschitz condition is satisfied, if f has a bounded partial derivative $\frac{\partial f}{\partial y}$ on G. Continuity of f is enough for existence, but not for uniqueness. The theorem does not guarantee global existence; in fact, this is very difficult topic. The name of the theorem is not as commonly used as others; the method of proof, called successive approximation or Picard iteration, has been used extensively by Picard[1], but the theorem and even method of proof had been known to others before him.

Any linear differential equation satisfies the conditions of Theorem 4.5; in fact, for linear differential equations one can establish global existence of solutions.

[1] Émile Picard (1856-1941), French mathematician is particularly well known for his contributions to the field of differential equations; in his other main area of interest, complex analysis, he contributed amongst other results a theorem called "Picard's Great Theorem", describing the behaviour of a complex function near an essential singularity

Example 4.6 *The IVP*

$$y' = 1 + y^2, \qquad y(0) = 0,$$

satisfies the conditions of the theorem, and

$$y(t) = \tan t$$

is the unique solution. But this solution does not continue beyond the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. *On the other hand, the IVP*

$$y' = \sqrt{y}, \qquad y(0) = 0,$$

does not satisfy a Lipschitz condition, although $f(t, y) = \sqrt{y}$ *is continuous. Indeed, solutions exist, but are not unique. Both functions*

$$y(t) \equiv 0 \qquad \textit{and} \qquad y(t) = \frac{t^2}{4}$$

solve the initial value problem.

For the remainder of these notes we shall simply assume our initial value problem has a unique solution over the interval of interest, and that the function $f(t, y)$ has as many continuous derivatives as required in the context.

4.3 Euler's method

We describe a simple numerical method, namely Euler's[2] method. Because of its low accuracy it is not a recommended method in practice, although it is used in the practical solution of partial differential equations.

Consider the initial value problem

$$y' = f(t, y), \qquad y(0) = y_0. \tag{4.17}$$

Of course, we cannot expect to compute the solution $y(t)$ of (4.17) for all values of t. So we define a discrete grid with constant grid spacing $h > 0$

$$t_k := kh, \qquad k = 0, 1, 2, \ldots$$

[2]Leonhard Euler (1707-1783), Swiss scientist and mathematician. His achievements and contributions are so numerous that it is impossible to even hint at some in a footnote. The publication of his collected works will require more than a 100 large volumes. In fact, many equations, formulas, theorems etc. had to be named for the first person to study them **after** Euler to avoid confusion. In the relation $e^{i\pi} + 1 = 0$ Euler linked together five of the most important mathematical constants. There is a saying that all calculus text books since the 18th century are essentially copies of Euler's calculus texts or copies of copies of Euler's texts. Euler was a man of broad culture, and besides being one of the foremost scientists, he was also well versed in several languages. He was not only prolific in his writings: he had 13 children.

and approximate the solution y at the grid points by the numbers y_k,

$$y_k \approx y(t_k)$$

(note that $y(\cdot)$ always denotes the exact solution).

Euler's method is based on the idea of the direction field (see Figure 4.1 and Figure 4.4): At a given point (t, y) march in the direction $f(t, y)$ given by the differential equation for a given finite step size h. With the initial value y_0 given by the initial condition of the IVP we simply get

$$y_{k+1} = y_k + hf(t_k, y_k). \tag{4.18}$$

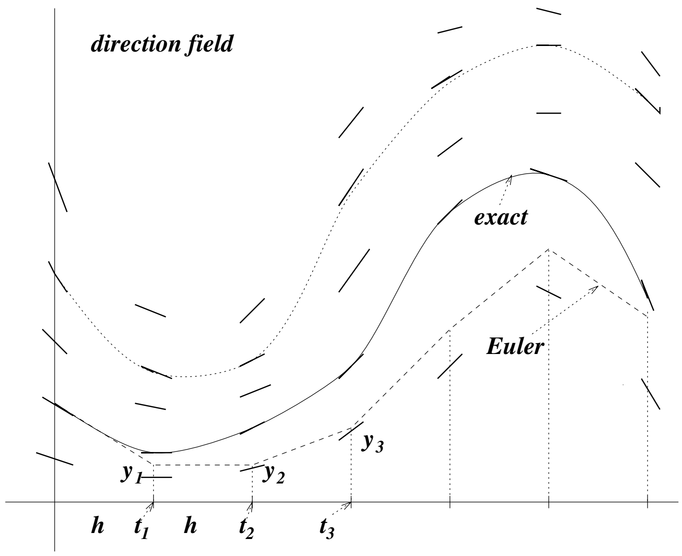

Figure 4.4: Direction field and Euler's method

Another way of justifying Euler's method is through a Taylor expansion:

$$\begin{aligned} y(t_k + h) &= y(t_k) + hy'(t_k) + \frac{h^2}{2} y''(t_k) + \dots \\ y(t_k + h) &= y(t_k) + hf(t_k, y(t_k)) + O(h^2). \end{aligned}$$