MATH 155 — Solutions to Quiz #1

February 09, 2000

Problem 1 We consider the function $g(x) = \tan x \cos x$ on the interval $(-\frac{\pi}{2}, +\frac{\pi}{2})$.

(a) Setting $u(x) = \tan x$ and $v'(x) = \cos x$ use **integration by parts** to find

$$\int \underbrace{\tan x}_{u} \underbrace{\cos x}_{v'} \, dx.$$

(b) Show an alternate way (not involving integration by parts) of obtaining the antiderivative of the function $\tan x \cos x$.

(a)

$$\begin{aligned}
\int \underbrace{\tan x}_{u} \underbrace{\cos x}_{v'} \, dx &= \underbrace{\tan x}_{u} \underbrace{\sin x}_{v} - \int \underbrace{\frac{1}{\cos^2 x}}_{u'} \underbrace{\sin x}_{v} \, dx && (1pt) \\
&= \tan x \sin x - \frac{1}{\cos x} + C = \frac{\sin^2 x}{\cos x} - \frac{1}{\cos x} + C && (2pts) \\
&= \frac{\sin^2 x - 1}{\cos x} + c = -\frac{\cos^2 x}{\cos x} + c = -\cos x + C. && (1pt)
\end{aligned}$$

The integral

$$\int \frac{\sin x}{\cos^2 x} \, dx$$

can be evaluated using the substitution $y = \cos x$, $dy = -\sin x dx$.

(b) (2 pts). Observe that $\tan x \cos x = \sin x$ on the given interval, hence

$$\int \tan x \cos x \, dx = \int \sin x \, dx = -\cos x + C.$$

Problem 2 Use **trigonometric substitution** to find

$$\int \frac{1}{\sqrt{4 - x^2}} \, dx.$$

The following formulas are used:

$$\begin{aligned}
x &= 2 \sin t && (1pt) \\
dx &= 2 \cos t dt && (1pt) \\
\sqrt{4 - x^2} &= \sqrt{4 - 4 \sin^2 t} = 2 \cos t. && (1pt)
\end{aligned}$$

This gives

$$\begin{aligned}
\int \frac{1}{\sqrt{4 - x^2}} \, dx &= \int \frac{2 \cos t}{2 \cos t} \, dt = \int 1 dt && (1pt) \\
&= t + C && (1pt) \\
&= \sin^{-1} \left(\frac{x}{2} \right) + C && (1pt).
\end{aligned}$$

Problem 3 Use **partial fractions** to find the antiderivative of

$$f(x) = \frac{x-8}{(x+1)(x-2)}$$

(a) Write down the partial fractions.

(b) Use the partial fractions to find $\int f(x)\,dx$.

(a)

$$\begin{aligned} \frac{x-8}{(x+1)(x-2)} &= \frac{A}{x+1} + \frac{B}{x-2} && (1pt) \\ (x-8) &= A(x-2) + B(x+1) \\ \text{Set } x=2:\ -6 &= 3B \qquad \Rightarrow B = -2, && (1pt) \\ \text{Set } x=-1:\ -9 &= -3A \qquad \Rightarrow A = 3. && (1pt) \end{aligned}$$

or solve system of two equations,

$$\begin{aligned} A + B &= 1 \\ -2A + B &= -8 \end{aligned}$$

giving the same solution $A = 3$, $B = -2$. Hence

$$\frac{x-8}{(x+1)(x-2)} = \frac{3}{x+1} - \frac{2}{x-2}.$$

(b)

$$\begin{aligned} \int \frac{x-8}{(x+1)(x-2)}\,dx &= 3\ln|x+1| - 2\ln|x-2| + C \\ &= \ln|x+1|^3 - \ln(x-2)^2 + C \\ &= \ln\frac{|x+1|^3}{(x-2)^2} + C. \end{aligned}$$

Any of these answers is o.k.; 3 points total. Deduct 1 point for missing constant C, deduct 1 pt for missing absolute value signs (where necessary).

Problem 4 Evaluate the following integrals. **2 points each; 1 point if answer is somewhat close to correct one.**

$$\begin{aligned} \frac{d}{dx}\left(\int_0^x e^{2t^2}\,dt\right) &= e^{2x^2}. \\ \int_1^x \frac{d}{dt}(t^3-1)\,dt &= \Big[t^3-1\Big]_1^x = x^3 - 1. \\ \frac{d}{dx}\left(\int_{\pi/4}^{\pi/2} \sin t\,dt\right) &= 0. \\ \frac{d}{dx}\left(\int_x^0 t^2\,dt\right) &= -x^2. \\ \int_0^1 \frac{d}{dx}\left((Tan^{-1}x)^2\right)dx &= \Big[(Tan^{-1}x)^2\Big]_0^1 = (Tan^{-1}(1))^2 - (Tan^{-1}(0))^2 = \left(\frac{\pi}{4}\right)^2 = \frac{\pi^2}{16}. \end{aligned}$$

Problem 5 Use **substitution** and **integration by parts** to find

$$\frac{1}{2}\int \cos\left(\sqrt{x}\right)\,dx.$$

Using the substitution

$$t^2 = x, \qquad (1pt), \qquad 2tdt = dx \qquad (1pt),$$

we obtain

$$\frac{1}{2}\int \cos\left(\sqrt{x}\right)\,dx = \int t\cos t\,dt. \qquad (1pt).$$

To evaluate this integral we use integration by parts,

$$\begin{aligned}\frac{1}{2}\int \cos\left(\sqrt{x}\right)\,dx = \int \underbrace{t}_{u}\,\underbrace{\cos t}_{v'}\,dt &= \underbrace{t}_{u}\,\underbrace{\sin t}_{v} - \int \underbrace{1}_{u'}\,\underbrace{\sin t}_{v}\,dt \qquad (1pt)\\ &= t\sin t + \cos t + C \qquad (1pt)\\ &= \sqrt{x}\sin\sqrt{x} + \cos\sqrt{x} + C. \qquad (1pt)\end{aligned}$$

Problem 6 Evaluate the following definite integrals

(a)

$$\int_{-2}^{+2}\left(4x^3 + 3x^2 - 101x - 4\right)\,dx =$$

(b)

$$\int_0^{2\pi}\cos^2 x\,dx + \int_0^{2\pi}\sin^2 x\,dx =$$

(a)

$$\begin{aligned}\int_{-2}^{+2}\left(4x^3 + 3x^2 - 101x - 4\right)\,dx &= \left[x^4 + x^3 - \frac{101}{2}x^2 - 4x\right]_{-2}^{+2} \qquad (2pts)\\ &= (16 + 8 - 202 - 8) - (16 - 8 - 202 + 8) = 0. \qquad (2pts)\end{aligned}$$

Using that x^3 and x are odd functions, the problem can be simplified by omitting these terms:

$$\begin{aligned}\int_{-2}^{+2}\left(4x^3 + 3x^2 - 101x - 4\right)\,dx &= \int_{-2}^{+2}\left(3x^2 - 4\right)\,dx\\ &= \left[x^3 - 4x\right]_{-2}^{+2} \qquad (2pts)\\ &= (8 - 8) - (-8 + 8) = 0. \qquad (2pts)\end{aligned}$$

(b) (2 points).

$$\int_0^{2\pi}\cos^2 x\,dx + \int_0^{2\pi}\sin^2 x\,dx = \int_0^{2\pi}(\cos^2 x + \sin^2 x)\,dx = \int_0^{2\pi}dx = 2\pi.$$

Alternatively, we can compute each of the integrals seperately. Using trigonometric identities

$$\cos^2 x = \frac{1}{2}(1 + \cos(2x)), \qquad \sin^2 x = \frac{1}{2}(1 - \cos(2x)),$$

we get

$$\begin{aligned}
\int_0^{2\pi} \cos^2 x \, dx + \int_0^{2\pi} \sin^2 x \, dx &= \int_0^{2\pi} \frac{1}{2}(1 + \cos(2x)) \, dx + \int_0^{2\pi} \frac{1}{2}(1 - \cos(2x)) \, dx \\
&= \left[\frac{x}{2} + \frac{1}{4} \sin(2x) \right]_0^{2\pi} + \left[\frac{x}{2} - \frac{1}{4} \sin(2x) \right]_0^{2\pi} \\
&= \pi + \pi = 2\pi.
\end{aligned}$$