```
Math 232, Jan 19, 2001:
  LU Factorization, switch rows ("pivoting"), zero pivots

A =
     2     1     1     1     1
    -4    -1    -1    -1    -1
     2     2     2     0     2
     6     5     2     2     4
     2     3     9    11     6
m=[2 -1 -3 -1];
P=eye(5); L=eye(5);
E1=eye(5); E1(2:5,1)=m'           %apply E1 to A to simultaneously create
E1 =                              %zeros in the first column
     1     0     0     0     0
     2     1     0     0     0
    -1     0     1     0     0
    -3     0     0     1     0
    -1     0     0     0     1
L(2:5,1)=-m'
L =
     1     0     0     0     0
    -2     1     0     0     0
     1     0     1     0     0
     3     0     0     1     0
     1     0     0     0     1
A1=E1*A
A1 =
     2     1     1     1     1
     0     1     1     1     1
     0     1     1    -1     1
     0     2    -1    -1     1
     0     2     8    10     5
m=[-1 -2 -2]; E2=eye(5);
E2(3:5,2)=m'
E2 =
     1     0     0     0     0
     0     1     0     0     0
     0    -1     1     0     0
     0    -2     0     1     0
     0    -2     0     0     1
L(3:5,2)=-m'
L =
     1     0     0     0     0
    -2     1     0     0     0
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     1     1     1     0     0
     3     2     0     1     0
     1     2     0     0     1
A2=E2*A1
A2 =
     2     1     1     1     1
     0     1     1     1     1
     0     0     0    -2     0
     0     0    -3    -3    -1
     0     0     6     8     3
                                        % In the (3,3) position, the next "pivot" is
                            % zero; switch rows 3 and 4 to get nonzero pivot
                            % ("pivoting"). Note the permutation matrix as a way to
                            % keep track of row changes in matrix language
P([3,4],:)=P([4,3],:)
P =
     1     0     0     0     0
     0     1     0     0     0
     0     0     0     1     0
     0     0     1     0     0
     0     0     0     0     1
A2=P*A2
A2 =
     2     1     1     1     1
     0     1     1     1     1
     0     0    -3    -3    -1
     0     0     0    -2     0
     0     0     6     8     3    %No we have nonzero element in pivot position
L =
     1     0     0     0     0
    -2     1     0     0     0
     1     1     1     0     0
     3     2     0     1     0
     1     2     0     0     1
L([3,4],1:2)=L([4,3],1:2)        % We must update the entries of L, as we
                                 % have switched rows 3 and 4
L =
     1     0     0     0     0
    -2     1     0     0     0
     3     2     1     0     0
     1     1     0     1     0
     1     2     0     0     1
A2
A2 =
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```
     2     1     1     1     1
     0     1     1     1     1
     0     0    -3    -3    -1
     0     0     0    -2     0
     0     0     6     8     3
E3=elemat(5,5,3,2)
E3 =
     1     0     0     0     0
     0     1     0     0     0
     0     0     1     0     0
     0     0     0     1     0
     0     0     2     0     1
L(5,3)=-2
L =
     1     0     0     0     0
    -2     1     0     0     0
     3     2     1     0     0
     1     1     0     1     0
     1     2    -2     0     1
A3=E3*A2
A3 =
     2     1     1     1     1
     0     1     1     1     1
     0     0    -3    -3    -1
     0     0     0    -2     0
     0     0     0     2     1
E4=elemat(5,5,4,1)
E4 =
     1     0     0     0     0
     0     1     0     0     0
     0     0     1     0     0
     0     0     0     1     0
     0     0     0     1     1
L(5,4)=-1;
U=E4*A3
U =
     2     1     1     1     1
     0     1     1     1     1
     0     0    -3    -3    -1
     0     0     0    -2     0
     0     0     0     0     1   %arrived at triangular form
L*U
ans =
     2     1     1     1     1
```

```
    -4    -1    -1    -1    -1
     6     5     2     2     4
     2     2     2     0     2
     2     3     9    11     6
A
A =
     2     1     1     1     1
    -4    -1    -1    -1    -1
     2     2     2     0     2
     6     5     2     2     4
     2     3     9    11     6
A-L*U
ans =
     0     0     0     0     0
     0     0     0     0     0
    -4    -3     0    -2    -2
     4     3     0     2     2
     0     0     0     0     0  % L*U does not give A -- we switched rows!
P*A
ans =
     2     1     1     1     1
    -4    -1    -1    -1    -1
     6     5     2     2     4
     2     2     2     0     2
     2     3     9    11     6
P*A-L*U
ans =
     0     0     0     0     0
     0     0     0     0     0
     0     0     0     0     0
     0     0     0     0     0
     0     0     0     0     0


%Now we factor a singular matrix  (precise definition of singular later;
%  for now:  singular matrix leads to zero pivots in the final U.
A =
     1     1     1     0
     1     1     2     1
     1     1     2     2
     1     1     3     0
L=eye(4); E1=eye(4);
m=[-1 -1 -1];
E1(2:4,1)=m'; L(2:4,1)=-m';
```

```
A1=E1*A
A1 =
     1     1     1     0
     0     0     1     1
     0     0     1     2
     0     0     2     0
m=[-1 -2];
E2=eye(4);E2(3:4,2)=m';
E2
E2 =
     1     0     0     0
     0     1     0     0
     0    -1     1     0
     0    -2     0     1
L(3:4,2)=-m',
L =
     1     0     0     0
     1     1     0     0
     1     1     1     0
     1     2     0     1
A2=E2*A1
A2 =
     1     1     1     0
     0     0     1     1
     0     0     0     1
     0     0     0    -2
E3=elemat(4,4,3,2);
L(4,3)=-2
L =
     1     0     0     0
     1     1     0     0
     1     1     1     0
     1     2    -2     1
U=E3*A2
U =
     1     1     1     0
     0     0     1     1
     0     0     0     1
     0     0     0     0
L
L =
     1     0     0     0
     1     1     0     0
     1     1     1     0
```

```
     1     2    -2     1

L*U
ans =
     1     1     1     0
     1     1     2     1
     1     1     2     2
     1     1     3     0
A
A =
     1     1     1     0
     1     1     2     1
     1     1     2     2
     1     1     3     0
U
U =
     1     1     1     0
     0     0     1     1
     0     0     0     1
     0     0     0     0

%This U has 0s on the diagonal.  It is in "staircase" form, or
%row-echelon form.
% Solving Ax = b is equivalent to solving Ux = c, where Lc=b.
```

We can only solve $Ux = c$ (i.e., $Ax = b$) when $c_4 = 0$, otherwise there is **NO SOLUTION!** If $c_4 = 0$ then:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$$

implies

$$x_4 = c_3 \qquad x_3 = c_2 - c_3 \qquad x_2 = t \qquad x_1 = c_1 - x_3 - x_2 = c_1 - c_2 + c_3 - t.$$

x_2 is called a free variable (it is not determined by the system), and t denotes a free parameter which can take any (real) value. In vector form:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} c_1 - c_2 + c_3 \\ 0 \\ c_2 - c_3 \\ c_3 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

The solutions form a "line in R^4", i.e., there are infinitely many solutions with one free parameter ("one-dimensional solution space").