Problem set #2 [Vector Geometry]

Due Wednesday, January 17

This problem set has review problems in analytical geometry. Make sure that you understand these problems, and that you are able to do them.

Equations of lines in the plane. A line in the plane $\boldsymbol{R}^2$ can be described in two different ways. Here is an example. Assuming you try to find the **equation** of the line through the points $(0,1)$ and $(2,5)$ in the (x,y)–plane, you will get $y = 2x+1$, or $-2x+y = 1$. There are several ways to compute this equation. Equivalently, you can give a representation of this line by a geometrically more intuitive procedure. Think of the given points as vectors. The direction of the line is given by

$$d = \begin{bmatrix} 2 \\ 5 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} .$$

We can therefore write the parameter representation of this line as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 4 \end{bmatrix} ,$$

where t is the parameter which is allowed to take any real value. Each point on the line corresponds to a specific value of t and vice versa. This equation means that to get to any point (vector) on the given line, we first go to one point on the line (here $(0,1)$), and then a certain distance (''t'') in the **direction** of the line. The direction vector is related to the slope of the line, which can be computed as the quotient of the y-component and x-component of the direction vector - in our example $4/2 = 2$. The representation of the line is by no means unique. For example,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} + u \begin{bmatrix} 1 \\ 2 \end{bmatrix} ,$$

represents the same line as above. Again, any point on the line corresponds to a specific value of u and vice versa.

Given the parameter representation of a line we can compute its equation by eliminating the parameter t (or u, or whatever name you choose for it). Computing the parameter representation from the equation corresponds to ''solving'' this equation, i.e., finding an explicit representation for the solution. So this is really only a special case of solving systems of equations, having one equation in two unknowns (x and y).

1) Find the equation and the parameter representation of the line

(a) going through (-1,0) and (2,3).

(b) going through (1,1) with slope 2.

(c) going through (-1,1) with slope $-1/2$.

(d) going through (2,1), (1,0), and (0,-1).

(e) going through (2,1), (1,0), and (-1,1).

(f) through (0,0) with direction vector $d = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

2) Find the point of intersection of the following two lines.

(a) The lines in 1a and 1b.

(b) The lines $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} + u \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(c) The lines given by $2x + y = 3$ and

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \end{bmatrix} .$$

(d) The lines given by $3x - 2y = 2$, and $4.5x - 3y = -1$.

Lines and planes in $\boldsymbol{R}^3$. The parameter representation of a line in three-dimensional space looks very much like the one in the two-dimensional case - except that all vectors now have three components:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} .$$

Eliminating t leaves us now with two equations in the three unknowns x, y, and z, the coordinates of a general point on the line:

$$y = -1 , \qquad 2x - z = 2 .$$

In $\boldsymbol{R}^3$ you can have lines which are not parallel, but do not intersect (this fact is rather helpful to air traffic controllers).

Given two points in $\boldsymbol{R}^3$, how can we construct the line going through these points? Let

$$p_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} , \qquad p_2 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} ; \qquad p_2 - p_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} ,$$

The *direction* vector of the line is then $p_2 - p_1$, and the parameter representation is simply

$$\mathbf{x} = p_1 + t\,(p_2 - p_1) ,$$

which yields our first example above.

The vector representation of a plane again follows similar ideas. However, on a plane we will have two directions (and any ''combination'' thereof) to move along. Here is an example:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + u \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + v \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} .$$

Eliminating the parameters u and v leaves us with only one equation in x, y, and z, describing the location of the points on our plane: $x - y + z = 2$ (this step takes a few lines of calculations which are omitted here).

3) Find the parameter representations of the following four lines (all lines are given by two points).

g_1: (0,–1,0), (2,1,2). g_3: (1,0,2), (–1,–2,0).

g_2: (2,–1,1), (0,1,1). g_4: (0,0,3), (3,0,0).

Find the point of intersection for each pair of lines. If the lines do not intersect, are they parallel, or not?

4) Find the parameter representation of the plane through the points (2,0,0), (1,1,2), and (1,–1,0). Convert this parameter form into the equation of the plane. Check your answer (the coordinates of the three given points must satisfy the equation). Also, find the points where the lines of problem 3) intersect the plane.

5) Find the parameter representation of the plane which is spanned by the two lines g_1 and g_2 of problem 3). Try the same for the lines g_2 and g_4.