MATH 416 — Problem set #2

Due, Wednesday, October 16, 2002, in class

Please note that Quiz #1 is on Monday, October 21. You are allowed to bring one letter-sized sheet (= two pages) with handwritten notes and an electronic calculator.

The first four problems are theoretical. Problem 5 requires some programming. You may use library routines as you please, but it will probably easier to write your own routines. Make your programs readable, but do not waste time on perfecting your output, etc.

Problem 1 Show by the method of undetermined coefficients that

$$u_{n+1} = -4u_n + 5u_{n-1} + h(4f_n + 2f_{n-1}) \tag{2.1}$$

is the most accurate explicit 2-step formula, with order of accuracy $p = 3$.

Problem 2 THEOREM: A linear multistep formula $< \rho, \sigma >$ has order of accuracy p if and only if

$$\rho(z) = \sigma(z)\, \log(z) + O\left((z-1)^{p+1}\right), \tag{2.2}$$

as $z \to 1$, with p taken as large as possible.

Consider the third order backward differentiation formula (cf. handout).

(a) What are $\rho(z)$ and $\sigma(z)$?

(b) Show that this formula is consistent.

(c) Use the Theorem above to show it has order of accuracy $p = 3$.

(d) Show that this formula is stable (i.e. zero-stable).

Problem 3 By looking at the model problem $u_t = qu$, $q < 0$, we found that Euler's method is conditionally stable for $h < -2/q$. Find the bound on h for the improved Euler method

$$u_{n+1} = u_n + \frac{h}{2}\left(f(t_n, u_n) + f(t_n + h, u_n + hf(t_n, u_n))\right).$$

(Optional: verify that for the classical Runge-Kutta method the condition is $|hq| < 2.785$).

Problem 4 For the two multistep methods below determine whether they are consistent, stable and/or convergent:

$$u_{n+1} - \frac{4}{3}u_n + \frac{1}{3}u_{n-1} = \frac{2h}{3}f_{n+1}. \tag{2.3}$$

$$u_{n+1} = u_n. \tag{2.4}$$

Problem 5 Consider a mathematical pendulum of length L and mass m. Denote by ϕ the angular deviation of the pendulum from the vertical position. The differential equation

$$\phi_{tt} = -\frac{g}{L} \sin \phi$$

with $g = 9.81 m s^{-2}$, describes the movement of the pendulum. In an elementary approach one replaces $\sin \phi$ by ϕ (a good linearization for small ϕ). The solutions are then harmonic oscillations with period $T = 2\pi\sqrt{L/g}$. Use a one-step numerical method of order at least 2 to find a better approximation to the period T, with initial position ϕ_0 and initial velocity zero. You will have to combine your DE-solver with a method to solve nonlinear equations (Newton works fine). Use $L = 1$ (meter), $phi_0 = \pi/3 = 60$deg. Start with an initial step size $h = 0.0125$, or even smaller. Compare your solution to the period predicted by the linear theory, and to the exact value ($T = 2.1529$). How well does the linear theory agree with your result when $\phi_0 = \pi/30 = 6$deg?