# MATH 416 — Problem set #3

Due, Friday, November 8, 2002

**Problem 1** Let $f$ be a 1-periodic function, with $f(x) = 1$ for $x \in (0, 1/2)$, and $f(x) = -1$ for $x \in (1/2, 1)$. Draw the graph of $f$, and compute its Fourier coefficients.

**Problem 2** Let $\delta$, $\mu$: $\ell_h^2 \rightarrow \ell_h^2$ be the discrete differentiation and smoothing operators defined by

$$(\delta v)_j := \frac{1}{2h}(v_{j+1} - v_{j-1}), \qquad (\mu v)_j := \frac{1}{2}(v_{j+1} + v_{j-1}).$$

(a) $\delta$ and $\mu$ can be expressed as convolutions with appropriate sequences $d$, $m \in \ell_h^2$. What are $d$ and $m$ ?

(b) Compute the Fourier transforms $\hat{d}$ and $\hat{m}$. Compare $\hat{d}$ to the transform of the exact differentiation operator for functions defined on $\mathbf{R}$. Draw a sketch.

(c) Compute $||d||$, $||\hat{d}||$, $||m||$, and $||\hat{m}||$, and verify Parseval's equality.

(d) Compute the Fourier transforms of the convolution sequences corresponding to the iterated operators $\delta^p$, and $\mu^p$ ($p \geq 2$). How do these results relate to the rule of thumb "the smoother the function, the narrower its Fourier transform"?

**Problem 3** *Numerical solution of the model problems.* The goal of this problem is to explore finite difference methods for the parabolic and hyperbolic (model) problems. In all parts below, your mesh should extend over an interval $[-M, +M]$ large enough to be "effectively infinite". For simplicity, impose the boundary conditions $u(-M, t) = u(M, t) = 0$. The initial function is $u_0(x) = \max(0, 1 - |x|)$ (the hat function extending over the interval $[-1, +1]$), and the computations should be carried until $t = T = 1$.

You will probably find it easiest to program all parts together (this does not mean "at once") in a single collection of subroutines, accepting various input parameters to control $h$, $k$, $M$, the choice of the finite difference formula, etc. Do **all** your computations for the space step $h$ being equal to negative powers of 2,

$$h = \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}.$$

Include plots when appropriate, but to see accuracy it may not be sufficient to just produce graphs of the solution profiles.

(a) Upwinding for $u_t = u_x$.

Write a program to solve $u_t = u_x$ by the UW formula with $k = h/2$, and with $k = 5h/4$. Plot the results (i.e. $v(x, 1)$), and comment on them (how do the results relate to our stability theory). Make a table of the computed values $v(-.5, 1)$, for each pair $(h, k)$. Estimate the order of accuracy from your experiment.

(b) Lax-Wendroff for $u_t = u_x$.

Repeat (a) for the LW formula, and for $k = 2h/5$.

(c) Euler for $u_t = u_{xx}$.

Extend the program to solve $u_t = u_{xx}$ by the $\mathrm{EU}_{xx}$ formula with

(i) $k = h/2$.

(ii) $k = 2h^2/5$.

Tabulate $v(1, 1)$ for each $(h, k)$.

(d) Tridiagonal system of equations.

Write a routine for solving a tridiagonal system of equations. We want to take advantage that we can solve such an $N$ by $N$ system in only $O(N)$ operations. The method to be employed is Gaussian elimination **without** pivoting. Test your routine carefully on some small systems whose exact solutions are known to you. If you prefer you may restrict your attention to the case of a symmetric positive definite matrix (this allows the use of a Cholesky factorization).

(e) Crank-Nicolson for $u_t = u_{xx}$.

Write down the tridiagonal matrix equation that has to be solved at each time step when the $\mathrm{CN}_{xx}$ formula is used. Apply your routine of (d) to carry out this computation with $k = h/2$. Make a table listing $v(1, 1)$ for each $h$, plot the results, and compare with (c).