MATH 416 — Problem set #4

Due, Friday, November 29, 2002

Problem 1 Dispersion relations. Consider a "normal mode" or "plane wave" solution

$$u(x,t) = e^{i(\omega t + \xi x)}$$

to a linear p.d.e. with constant coefficients that is first order in t. The wave number ξ is real, but the frequency ω maybe real or complex. The p.d.e. will impose a relationship

$$\omega = \omega(\xi)$$

between ω and ξ, known as the dispersion relation of the p.d.e. Determine the relation for the two model problems

$$\text{(a)} \qquad u_t = u_x, \qquad\qquad \text{(b)} \qquad u_t = u_{xx}.$$

Also compute this dispersion relationship for

(c) LW_x, the Lax-Wendroff scheme for $u_t = u_x$.

Problem 2 Amplification factors. Calculate the amplification factors for EU_x, BE_x, BOX_x, LXF_x, EU_{xx}, and BE_{xx}.

See the notes for the various finite difference formulas.

Problem 3 Generalized Crank-Nicolson.

Consider the heat equation $u_t = u_{xx}$ and the finite difference formula

$$\delta^+ v = (1-\theta)\delta_\times v + \theta \delta_\times Z v \tag{4.1}$$

with $0 \le \theta \le 1$. For $\theta = 1/2$ this is CN_{xx}, for $\theta = 0$ it is EU_{xx}. (Z is the time-shift operator, δ^+ the forward difference operator in time, and $\delta_\times$ is the spatial symmetric difference operator approximating the second derivative, as all defined in the lecture notes).

(a) Compute the amplification factor function $g(\xi)$.

(b) Let $\sigma = k/h^2$ be constant as $k \to 0$. For which σ and θ is (GCN) stable?

(c) Let $\lambda = k/h$ be constant as $k \to 0$. For which λ and θ is (GCN) stable?