James T. Sandefur, *Discrete Dynamical Systems: Theory and Applications*, Clarendon Press, Oxford, 1990. Sandefur has a second book on discrete dynamical modeling, but we prefer this one.

A comprehensive though terse book for further study in both discrete and continuous models:

J.D. Murray, *Mathematical Biology*, Springer-Verlag, Berlin, 1989.

1.8 Exercises

1. Use an induction proof to verify that $x(n) = R^n x(0)$ is a solution of $x(n) = Rx(n-1)$.

2. Find a closed-form solution for the affine recurrence relation

$$x(n) = Rx(n-1) + a.$$

3. Find fixed points for the following recursion relations, and test for stability. Draw a cobweb diagram for parts c and d.

a. $x(n) = \dfrac{x(n-1)}{1 + x(n-1)}$.
b. $x(n) = x(n-1)e^{r\,x(n-1)}$ where r is a constant.
c. $x(n) = x(n-1)^2 - 6$.
d. $x(n) = x(n-1)^2 + .7x(n-1) + .02$

4. Construct a recursion relation that has a stable fixed point at $x = 1$ and an unstable fixed point at $x = 3$.

1.9 Projects

Project 1.1. The Credit Card Problem

The motivation for this project comes from the following problem which is paraphrased from a calculus book.

> A man has a credit card with a $1.000 balance, 18.9% annual interest, and no grace period. He makes no payments for a year and is surprised when his statement shows a balance of more than $1.189. Use the idea of monthly compounding to help him out.

This project starts with this simple problem and successively refines it to incorporate payment schemes (a credit card with no payments for a year?) and variable interest rates.

1. Start with the compartmental diagram shown in Figure 1.24. Assuming an initial $1,000 balance, 18.9% annual interest rate, monthly compounding, no grace period and no payments, determine the balance after one year and five years.

2. Introduce payments to the above model by adding a flow out of the balance box. Assume a minimum monthly payment of 3% of the balance, determine what the balance will be after five years. How long will it take to get the balance down to under 10 dollars?

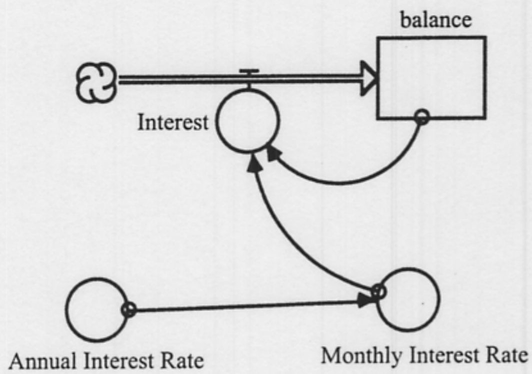

FIGURE 1.24 Credit Card Compartmental Diagram

3. One of the authors, has a credit card which has a minimum payment of 3% of the balance or \$10, whichever is larger. With this assumption, how long will it take to pay off the credit card? Note: When defining payments you may use an IF THEN ELSE construction.

4. Keeping with the assumptions in 3, suppose we are interested in how much borrowing \$1,000 costs. Add a mechanism for keeping track of total payments. A compartmental diagram is shown in Figure 1.25.

5. Suppose we pay \$10 more than the minimum payment each month. What effect does this have on the total payments and the time to pay off the card?

Project 1.2. Bobcats

Most species of wild cats are endangered including the bobcat. This project explores the behavior of a bobcat population using growth rate data from the state of Florida (from Cox, et. al., *Closing the Gaps in Florida's Wildlife Habitat Conservation System* [13]). We know annual growth rates for bobcats under best ($r = 0.01676$), medium ($r = 0.00549$), and worst ($r = -0.04500$) environmental conditions. For this project,

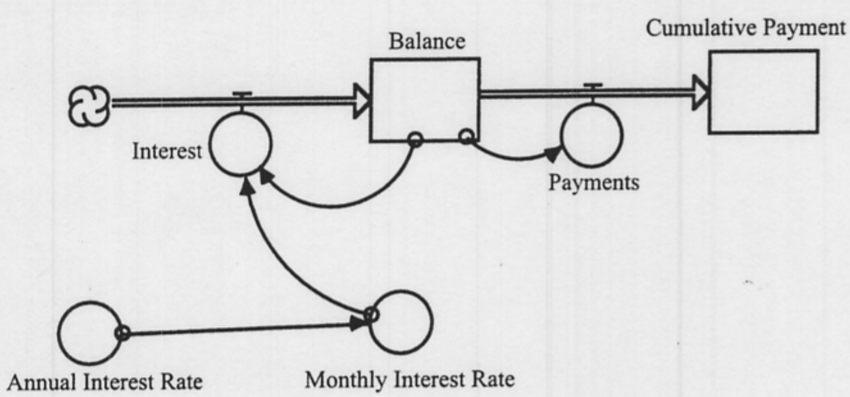

FIGURE 1.25 A More Refined Model

assume that growth rates are constant from year to year (complications will come later). For now consider each growth rate as representing a different region in Florida.

1. Construct a spreadsheet (or other simulation) tracking three bobcat populations, each initially consisting of 100 individuals, over a period of 10 years under the three types of environmental conditions. Plot all three simulations on a single graph. The graph should have dates on the horizontal axis (starting with the current year), it should have a legend identifying which curve is which, and the style should have each curve identifiable when printed in black ink.

2. Repeat part **1** over a period of 25 years.

3. You should have noticed that under the best conditions the population is growing. Several management plans have been discussed. The first is to allow one bobcat per year to be hunted. The second is to allow five bobcats per year to be hunted. The third is to allow one percent of the animals to be hunted. The last is to let five percent of the animals be hunted. Construct a simulation which compares these strategies over 10 and 25 years. Which of these strategies result in a stable population?

4. Continuing with the theme of part **3**, experiment to find strategies which cause the population to stabilize. Find a strategy (which can be more involved than just subtracting a fixed number or percentage) which causes the population to rise to 200 animals (give or take an animal or two) and then stabilize.

5. You should also have noticed that under the worst conditions the population is declining. Proposed management plans include adding three animals per year, adding 10 animals per year, adding one percent of the population each year, and adding five percent of the population each year. Compare these strategies for 10 and 25 years. Which strategies cause the population to stabilize?

6. Experiment to find strategies which will cause the population under the worst conditions to stabilize at 50 and at 200 animals.

7. Which of the strategies in parts **3** and **5** are affine? By hand, analyze these equations (which are affine) to find fixed points and test for stability. Use the theory of affine equations to verify the numerical results.

Project 1.3. Blood Alcohol Model

Data and Assumptions: (Data are from Davis, Porta, Uhl, *Calculus&Mathematica: Derivatives*[14] ©1994 Addison Wesley Longman Inc. Reprinted by Permission of Addison Wesley Longman.)

The average human body eliminates 12 grams of alcohol per hour.

An average college age male in good shape weighing K kilograms has about $.68K$ liters of fluid in his body. A college-age female in good shape weighing K kilograms has about $.65K$ liters of fluid in her body. People in poor shape have less.

One kilogram = 2.2046 pounds.

Threshold for legal driving: If your body fluids contain more than one gram of alcohol per liter of body fluids (or 0.1 gm/100 ml which is the usual way of reporting it),

Type of Drink	Grams of Alcohol
12 ounce regular beer	13.6
12 ounce light beer	11.3
4 ounce port wine	16.4
4 ounce burgundy wine	10.9
4 ounce rose wine	10.0
1.5 ounce 100-proof vodka	16.7
1.5 ounce 100-proof bourbon	16.7
1.5 ounce 80-proof vodka	13.4
1.5 ounce 80-proof bourbon	13.4

TABLE 1.1 Grams of Alcohol for Different Types of Drinks.

then you are too drunk to drive legally in most states. Find out the level for your state and use it in this project.

A blood alcohol concentration of 4.0 gm/l is likely to result in coma. A blood alcohol concentration of 4.5-5.0 gm/l is likely to result in death.

Alcohol content of various beverages: see Table 1.1

Project: Construct the basic model from the compartmental diagram shown in Figure 1.26. This diagram was drawn using Stella; recall that circles either hold parameters or perform operations on parameters. Pick an appropriate time step. We suggest one minute. (Technically this is a continuous model, but we are treating it as discrete with a short time step.) Pick a hypothetical weight and sex.

1. Assume your hypothetical person arrives at a party and instantaneously downs a six-pack of beer (i.e., high initially alcohol level, no input flow). Graph alcohol concentration

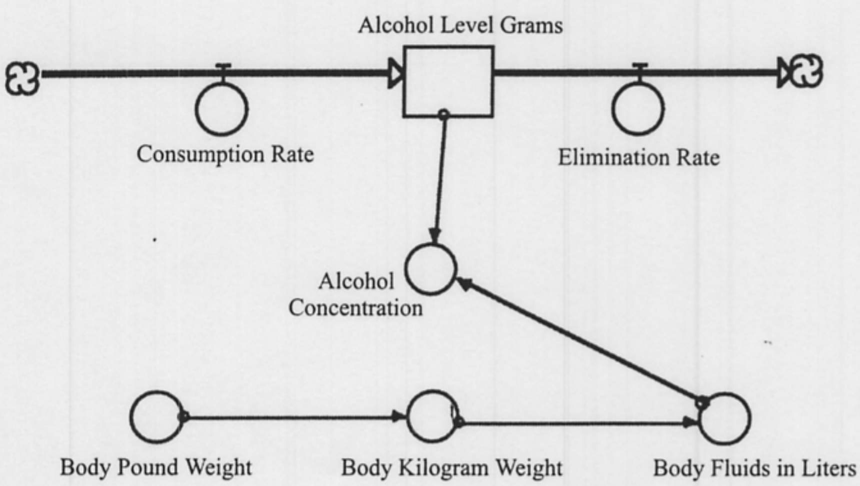

FIGURE 1.26 Compartmental Diagram for the Blood Alcohol Model.

as a function of time. Plot the legal driving level on the same graph. Be very careful of the scales. How long will it be before Hypothetical can drive home legally?

2. Construct a more realistic manner of consuming six beers (i.e., zero initial alcohol level, positive flow). You may want to use if-then statements to turn off the drinking after a time, or to change rates of drinking throughout the evening. Again plot alcohol concentration against time and include the legal driving level on the graph.

3. Try some other alcohol input functions. Here are some ideas. A piecewise defined function can be used to model periods of drinking and non-drinking. A function that steps up for a short time, then back down (some programs even have a special pulse function) can be used for simulating shots. One could construct a function which relates the rate of drinking to alcohol concentration or to the number of people at a party.

Graphs should include the legal driving, coma, and death levels.

Prepare an information sheet or sheets for distribution to students (your peers) showing the effects of drinking styles on blood alcohol concentration.

We acknowledge a high school teacher in the TORCH program, whose name we no longer recall, for originally introducing us to the ideas of this project. The book *Calculus&Mathematica* by Davis, Porta, and Uhl has a problem on blood alcohol levels which significantly influenced this project.

Project 1.4. Compound Interest

When you save money in a bank account, interest is compounded periodically, which means that periodically the interest you earn is added to your account so that you start earning interest on your interest.

1. Suppose you deposit $\$1,000$ into a bank account that has a 5% annual interest rate. Further suppose that interest is compounded monthly. Thus initially you have \$1,000; at the end of one month your account has $\$1{,}000 + 0.05/12 \times 1{,}000$ or $\$1{,}004.17$; at the end of two months your account has $\$1{,}004.17 + 0.05/12 \times 1{,}004.17$ or $\$1{,}008.35$; and so on. Notice that even though the model's scale is in years, the time step is one month. Notice further that since the interest rate is given in years (the time scale), we need to divide by 12 to compute the interest rate per time step. Set up a recurrence relation for the amount of money $x(t+1)$, given that we know the amount of money in the account after t time steps $x(t)$. Track the amount of money in the account over two years.

2. Find a closed form solution for $x(i)$, the amount in the account after i time steps, using data and assumptions from part **1**.

3. Find the recurrence relation for the amount of money in the account x if P dollars are invested at an annual interest rate r which is compounded n times per year for i time steps.

4. Solve to find a closed-form solution for the recurrence relation in part **3**. Express the function in terms of t years, instead of i time steps.

5. Suppose now that you deposit $\$1.000$ in an account that is compounded monthly but which has a variable interest rate. Track your account over two years using the monthly interest rates given in Table 1.2. The rate at $t = 0$ corresponds to the rate from the moment of deposit until the first compounding period.

Month	Rate	Month	Rate
0	0.05	13	0.049
1	0.051	14	0.049
2	0.052	15	0.048
3	0.053	16	0.048
4	0.052	17	0.048
5	0.052	18	0.047
6	0.053	19	0.048
7	0.053	20	0.047
8	0.053	21	0.046
9	0.052	22	0.045
10	0.052	23	0.045
11	0.051	24	0.044
12	0.051	25	0.044

TABLE 1.2 Monthly Annual Interest Rates.

Project 1.5. Inflation

You should work through Project 1.4 before working this problem. Both terminology and results of Project 1.4 are needed. As you are probably aware, inflation is the annual increase in the prices of goods and services. Suppose inflation raises prices this year by 3% over last year, and next year inflation raises prices 3% over this year's prices. Notice that each year the inflation is raising not the original price, but the inflated price. This is compound interest compounded annually with a slight perspective shift.

1. Assuming an annual inflation rate of 3%, how much will an item that currently costs $100 cost in 1, 5, 10, and 30 years? (Assume a compound interest compounded annually type model. Some items obey this, others do not. We are assuming this one does.)

2. Still assuming that an annual inflation rate of 3% held for the past 10 years, how much did an item that costs $100 today cost 10 years ago? (Again assume a compound-interest-compounded-annually-type model.)

3. Look up the annual inflation rate for the past 30 years. (One source is the Consumer Price Index in the Statistical Abstract of the United States published by the Census Bureau; online versions exist.) If an item cost $100 thirty years ago, how much did it cost after 1, 5, 10, and 30 (today) years? What is the average inflation rate over this time period. Assuming a constant annual inflation rate equal to the 30-year average, how much did a $100 item cost after 30 years? Compare your answers from the two models. You should address the effect of using an average value instead of the actual value. Once again figure the cost of a $100 item after 30 years assuming the inflation rate of 30 years ago remained constant. Compare this result with the results of the first two models. How good were the predictions? This is essentially what we do when we plan for our retirement using current inflation rates. Thirty years from now, we will see how good our choice of inflation rate was.

Project 1.6. Plant-Herbivore System

The purpose of this project is to model the dynamics involved in introducing a population of 100 deer into an area consisting of a patchwork of grassland and forest. By "dynamics" we mean we want to predict the type of behavior seen (growing, declining, oscillating, chaotic), and we are not interested in predicting actual numbers of animals in a given year. Rather than count individual plants, we measure the plant density in appropriate density units. In the absence of deer, the plants observe a logistic type growth with a growth rate of 0.8 when the plant density is low and a growth rate near 0 when the plant density is near 3,000 density units. Initially the area is ungrazed, so we assume that the initial plant density is 3,000 units. The deer herd eats the plants, of course, and when the vegetation is burned out or grazed flat, the herd will decline at a rate of 1.1. (That is, the number of deer lost in a year is 1.1 times the number of deer. This seems impossible, but it only means that the herd would be gone in less than a year with no food.) When food is present, this decline is offset by a growth term which has a rate of 0 when plant density is 0, 1.5 when plant density is 3.000, and is roughly linear (assume it is linear) in between. (Note this is the growth rate, not the actual number of deer.) Similarly when deer are present, the plant density is diminished. Each deer can eat as much as 1.2 density units per year when food is plentiful (plant density of 3,000), but as food becomes scarcer, the deer will eat less with no food eaten when there is no food available. Again assume a linear relationship between eating no food when the plant density is zero and 1.2 density units when the density is 3,000.

Part I

1. Construct a simulation model and run it for 30 years under the following three conditions

a. Discrete time steps of one year

b. Discrete time steps of one day

c. Optional (If the software that is being used for the simulation easily does this; otherwise it will be discussed in Chapter 5): Continuous time (fractional time steps)

2. For each of the cases in **1**, make a graph with the deer and plant density plotted as functions of time on the same axis (the scales may be different). Also make a phase plane plot for each case (deer on one axis and plants on the other). Include comments about existence and stability of fixed points and the differences between the various cases. Which case is most appropriate for this scenario?

Part II

The situation above is a modification of a model by Caughley which is identical except that the interaction terms for the deer and plants do not vary linearly. His model had the decrease in plant density due to deer equal to 1.2 deer population $(1 - \exp(-.001 *$ plant density$))$, and the increase in deer due to the plants as 1.5 deer population$*(1 - \exp(-.001 *$ plant density$))$.

Replace the plant-deer interaction terms with Caughley's terms and repeat the work of Part I, including the discussion.

Part III

Compare the models in Part I and Part II. Part of this comparison should involve plotting (1-exp(-.001 x)) from 0 to 5.000 and explaining what this term contributes to the second model over the line used in the first model. Which model does the best job of explaining the observed behavior?